Math 322

March 14, 2022

stable, unstable, and center subspaces

$A \in M_n(\mathbb{R}^n)$ with generalized eigenvectors and eigenvalues

$$u_j + iv_j \in \mathbb{C}^n \quad \text{and} \quad \lambda_j = a_j + ib_j \in \mathbb{C},$$

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respectively, for $j = 1, \ldots, n$.

$$\begin{aligned} E^s &:= \operatorname{Span}\{u_j, v_j : a_j < 0\} \\ E^u &:= \operatorname{Span}\{u_j, v_j : a_j > 0\} \\ E^c &:= \operatorname{Span}\{u_j, v_j : a_j = 0\}. \end{aligned}$$

Stable manifold theorem

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such that

$$\lim_{t \to \infty} \phi_t(p) = 0$$

for any $p \in S$ and

$$\lim_{t \to -\infty} \phi(p) = 0$$

for any $p \in U$.

Example

$$x' = -x - y^2$$
$$y' = y + x^2$$

Sketch of proof of stable manifold theorem

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(iii) Take P so that

$$P^{-1}Jf(0)P = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

where A has k eigenvalues with positive real part and B has $n-k$ eigenvalues with negative real part.

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(iv) Let $y = P^{-1}x$ and $G(y) = P^{-1}F(Py)$:

$$y' = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} y + G(y).$$

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operator on functions $u\colon [-\varepsilon, \varepsilon] \times \Omega \to \mathbb{R}^n$:

$$(Tu)(t,a) = U(t)a + \int_{s=0}^{t} U(t-s)G(u(s,a))\,ds - \int_{s=t}^{\infty} V(t-s)G(u(s,a))\,ds$$

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idea: iterate T starting at the zero-function

Sketch of proof of stable manifold theorem

Fixed point u of T:

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The stable manifold is the set of points

$$(a_1,\dots,a_k,u_{k+1}(0,a_1,\dots,a_k,0,\dots,0),\dots,u_n(0,a_1,\dots,a_k,0,\dots,0))$$

as $(a_1,\dots,a_k)$ varies in a neighborhood of the origin in $\mathbb{R}^k$

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Apply $\phi\colon y \mapsto (y_{k+1}, \ldots, y_n, y_1, \ldots, y_k)$:

$$(\phi(y))' = \begin{pmatrix} -B & 0 \\ 0 & -A \end{pmatrix} \phi(y) - G(\phi(y))$$

Find the stable manifold for this system, then apply ϕ^{-1}.

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&\Rightarrow \quad y(t) - e^{Mt}y(0) = \int_{s=0}^{t} e^{M(t-s)}G(y(s))\,ds
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