

Math 322

March 18, 2022

During break

- ▶ Think about projects.
- ▶ Homework due on Monday, March 28.

Stable manifold theorem

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such that

$$\lim_{t \rightarrow \infty} \phi_t(p) = 0$$

for any $p \in S$ and

$$\lim_{t \rightarrow -\infty} \phi_t(p) = 0$$

for any $p \in U$.

Global stable and unstable manifolds

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In addition to the global stable and unstable manifolds, there also exists a global *center manifold* tangent to the center space of the linearized system and invariant under flow.

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Proof. Successive approximations. See text.

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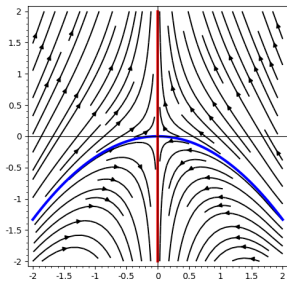
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Check that H preserves solutions and the (un)stable manifolds.

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H →

