Math 322

March 18, 2022

During break

- Think about projects.
- Homework due on Monday, March 28.

Stable manifold theorem

Theorem. $f \in C^1(E)$,

Stable manifold theorem

Theorem. $f \in C^1(E)$, $f(0) = 0$,

Stable manifold theorem

Theorem. $f \in C^1(E)$, $f(0) = 0$,

Df_0: k eigenvalues with negative real part and $n - k$ eigenvalues with positive real part.

Stable manifold theorem

Theorem. $f \in C^1(E)$, $f(0) = 0$,

Df_0: k eigenvalues with negative real part and $n - k$ eigenvalues with positive real part.

$\exists$ k-dimensional differentiable manifold S tangent to the stable subspace E^s of the linearized system $x' = Df_0(x)$ at 0

Stable manifold theorem

Theorem. $f \in C^1(E)$, $f(0) = 0$,

Df_0: k eigenvalues with negative real part and $n - k$ eigenvalues with positive real part.

$\exists$ k-dimensional differentiable manifold S tangent to the stable subspace E^s of the linearized system $x' = Df_0(x)$ at 0

and $\exists$ $(n-k)$-dimensional differentiable manifold U tangent to the unstable space E^u of the linearized system

Stable manifold theorem

Theorem. $f \in C^1(E)$, $f(0) = 0$,

Df_0: k eigenvalues with negative real part and $n - k$ eigenvalues with positive real part.

$\exists$ k-dimensional differentiable manifold S tangent to the stable subspace E^s of the linearized system $x' = Df_0(x)$ at 0

and $\exists$ $(n-k)$-dimensional differentiable manifold U tangent to the unstable space E^u of the linearized system

such that

$$\lim_{t \to \infty} \phi_t(p) = 0$$

for any $p \in S$ and

$$\lim_{t \to -\infty} \phi(p) = 0$$

for any $p \in U$.

Global stable and unstable manifolds

Define

$$W^s(0) := \cup_{t \leq 0} \phi_t(S)$$

and

$$W^u(0) := \cup_{t \geq 0} \phi_t(U).$$

Global stable and unstable manifolds

Define

$$W^s(0) := \cup_{t \leq 0} \phi_t(S)$$

and

$$W^u(0) := \cup_{t \geq 0} \phi_t(U).$$

These manifolds

Global stable and unstable manifolds

Define

$$W^s(0) := \cup_{t \leq 0} \phi_t(S)$$

and

$$W^u(0) := \cup_{t \geq 0} \phi_t(U).$$

These manifolds (i) do not depend on our choice of local stable and unstable manifolds S and U,

Global stable and unstable manifolds

Define

$$W^s(0) := \cup_{t \leq 0} \phi_t(S)$$

and

$$W^u(0) := \cup_{t \geq 0} \phi_t(U).$$

These manifolds (i) do not depend on our choice of local stable and unstable manifolds S and U, (ii) are invariant under ϕ_t,

Global stable and unstable manifolds

Define

$$W^s(0) := \cup_{t \leq 0} \phi_t(S)$$

and

$$W^u(0) := \cup_{t \geq 0} \phi_t(U).$$

These manifolds (i) do not depend on our choice of local stable and unstable manifolds S and U, (ii) are invariant under ϕ_t, and (iii) for all $p \in W^s(0)$,

$$\lim_{t \to \infty} \phi_t(p) = 0$$

and for all $p \in W^u(0)$,

$$\lim_{t \to -\infty} \phi_t(p) = 0.$$

Center manifold theorem

Theorem. $f \in C^1(E)$,

Center manifold theorem

Theorem. $f \in C^1(E)$, $f(0) = 0$,

Center manifold theorem

Theorem. $f \in C^1(E)$, $f(0) = 0$,

Df_0: k eigenvalues with negative real part, j eigenvalues with positive real part, and $n - k - j$ eigenvalues with zero real part

Center manifold theorem

Theorem. $f \in C^1(E)$, $f(0) = 0$,

Df_0: k eigenvalues with negative real part, j eigenvalues with positive real part, and $n - k - j$ eigenvalues with zero real part

In addition to the global stable and unstable manifolds, there also exists a global *center manifold* tangent to the center space of the linearized system and invariant under flow.

Hartman-Grobman theorem

$f \in C^1(E)$, $f(0) = 0$, $Jf(0)$ has k eigenvalues with positive real part and $n - k$ with negative real part

Hartman-Grobman theorem

$f \in C^1(E)$, $f(0) = 0$, $Jf(0)$ has k eigenvalues with positive real part and $n - k$ with negative real part (i.e., the origin is a *hyperbolic equilibrium* point for the system)

Hartman-Grobman theorem

$f \in C^1(E)$, $f(0) = 0$, $Jf(0)$ has k eigenvalues with positive real part and $n - k$ with negative real part (i.e., the origin is a *hyperbolic equilibrium* point for the system)

Theorem. There exist neighborhoods of the origin U and V and a homeomorphism $H\colon U \to V$

Hartman-Grobman theorem

$f \in C^1(E)$, $f(0) = 0$, $Jf(0)$ has k eigenvalues with positive real part and $n - k$ with negative real part (i.e., the origin is a *hyperbolic equilibrium* point for the system)

Theorem. There exist neighborhoods of the origin U and V and a homeomorphism $H\colon U \to V$ such that for all $x_0 \in U$, there is an interval I containing the origin and

Hartman-Grobman theorem

$f \in C^1(E)$, $f(0) = 0$, $Jf(0)$ has k eigenvalues with positive real part and $n - k$ with negative real part (i.e., the origin is a *hyperbolic equilibrium* point for the system)

Theorem. There exist neighborhoods of the origin U and V and a homeomorphism $H\colon U \to V$ such that for all $x_0 \in U$, there is an interval I containing the origin and

$$H(\phi_t(x_0)) = e^{Jf(0)t} H(x_0).$$

Hartman-Grobman theorem

$f \in C^1(E)$, $f(0) = 0$, $Jf(0)$ has k eigenvalues with positive real part and $n - k$ with negative real part (i.e., the origin is a *hyperbolic equilibrium* point for the system)

Theorem. There exist neighborhoods of the origin U and V and a homeomorphism $H\colon U \to V$ such that for all $x_0 \in U$, there is an interval I containing the origin and

$$H(\phi_t(x_0)) = e^{Jf(0)t} H(x_0).$$

Proof. Successive approximations. See text.

Example

$$x' = -x$$
$$y' = y + x^2$$

Example

$$x' = -x$$
$$y' = y + x^2$$

Solution with initial condition (x_0, y_0):

$$x(t) = x_0 e^{-t}$$

$$y(t) = \left(y_0 + \frac{1}{3}x_0^2\right) e^t - \frac{1}{3}x_0^2 e^{-2t}.$$

Example

$$x' = -x$$
$$y' = y + x^2$$

Solution with initial condition (x_0, y_0):

$$x(t) = x_0 e^{-t}$$

$$y(t) = \left(y_0 + \frac{1}{3}x_0^2\right) e^t - \frac{1}{3}x_0^2 e^{-2t}.$$

Homeomorphism:

$$H(x, y) = \left(x, y + \frac{1}{3}x^2\right).$$

Example

$$x' = -x$$
$$y' = y + x^2$$

Solution with initial condition (x_0, y_0):

$$x(t) = x_0 e^{-t}$$

$$y(t) = \left(y_0 + \frac{1}{3}x_0^2\right) e^t - \frac{1}{3}x_0^2 e^{-2t}.$$

Homeomorphism:

$$H(x, y) = \left(x, y + \frac{1}{3}x^2\right).$$

Check that H preserves solutions and the (un)stable manifolds.

Example