Math 322

March 7, 2022

Derivatives

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approximating f near p:

$$f(p+h) \approx f(p) + Df_p(h)$$

for small h.

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Definition. The function $f\colon E \to \mathbb{R}^n$ is *continuously differentiable* if

$$\begin{aligned} E &\to \mathcal{L}(\mathbb{R}^n) \\ p &\mapsto Df_p \end{aligned}$$

is continuous.

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$$\frac{\partial f}{\partial x_j}(p) = \begin{pmatrix} \frac{\partial f_1}{\partial x_j}(p) \\ \frac{\partial f_2}{\partial x_j}(p) \\ \vdots \\ \frac{\partial f_n}{\partial x_j}(p) \end{pmatrix}.$$

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Notation. The space of continuously differentiable functions $E \to \mathcal{L}(\mathbb{R}^n)$ is denoted $C^1(E)$.

Lipschitz condition

Definition. Let $E \subseteq \mathbb{R}^n$ be an open subset. Then a function $f\colon E \to \mathbb{R}^n$ is *Lipschitz* if there exists a constant K such that

$$|f(x) - f(y)| \leq K|x - y|$$

for all $x, y \in E$.

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The function f is *locally Lipschitz* on E if for each $x_0 \in E$, there exists $\varepsilon > 0$ and a constant K such that

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Proposition. If $f \in C^1(E)$, then f is locally Lipschitz.

Fundamental existence and uniqueness theorem

Theorem. Let E be an open subset of $\mathbb{R}^n$ containing x_0, and let $f \in C^1(E)$. Then there exists $a > 0$ such that the initial value problem

$$\begin{aligned} x' &= f(x) \\ x(0) &= x_0 \end{aligned}$$

has a unique solution $x(t)$ on $[-a, a]$.

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Method of successive approximations:

$$\begin{aligned} T\colon C(I) &\to C(I) \\ u &\mapsto x_0 + \int_{s=0}^{t} f(u(s))\, ds. \end{aligned}$$