Math 322

March 11, 2022

Dependence on parameters

Theorem. Let E be an open subset of $\mathbb{R}^{n+m}$ containing the point (x_0, μ_0) where $x_0 \in \mathbb{R}^n$ and $\mu_0 \in \mathbb{R}^m$, and assume $f \in C^1(E)$.

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$$\begin{aligned} x' &= f(x, \mu) \\ x(0) &= y \end{aligned}$$

has a unique solution $x = x(t, y, \mu)$ with $x \in C^1(R)$ where $R := [-a, a] \times N(x_0) \times N(\mu_0)$.

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Stable manifold theorem

Theorem. (Stable manifold theorem.) Let $E \subseteq \mathbb{R}^n$ and let $f \in C^1(E)$. Suppose that $f(0) = 0$ and that Df_0 has k eigenvalues with negative real part and $n - k$ eigenvalues with positive real part.

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Then there exists a k-dimensional differentiable manifold S tangent to the stable subspace E^s of the linearized system $x' = Df_0(x)$ at 0, and there exists an $(n-k)$-dimensional differentiable manifold U tangent to the unstable space E^u of the linearized system with the properties

$$\lim_{t \to \infty} \phi_t(x_0) = 0$$

for any $x_0 \in S$ and

$$\lim_{t \to -\infty} \phi(x_0) = 0$$

for any $x_0 \in U$.

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Metric space

Definition. A *metric space* is a set X with a *distance function* or *metric*,

$$d\colon X \times X \to \mathbb{R}$$

that is positive definite, symmetric, and obeys the triangle inequality:

1. $d(x, y) \geq 0$ with $d(x, y) = 0$ if and only if $x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, y) \leq d(x, z) + d(z, y)$.

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Every metric space (X, d) is a topological space where a subset $U \subseteq X$ is *open* if for each $u \in U$, there exists $r > 0$ such that the open ball of radius r centered at u is contained in U:

$$B(u, r) := \{x \in X : d(u, x) < r\} \subseteq U.$$

Manifold

Definition. An n-dimensional differentiable manifold is a connected metric space[2] M and an open covering $\{U_\alpha\}$ (so for each α in some index set, U_α is an open subset of M and $M = \cup_\alpha U_\alpha$) such that:

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Each pair (h_α, U_α) is called a *chart*, and the collection of charts is called an *atlas*. The mapping $h_\beta \cap h_\alpha^{-1}$ are *transition functions*.

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