Math 322

March 2, 2022

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Solution: An interval I containing t_0 and a parametrized curve $x: I \to E \subseteq \mathbb{R}^n$ with x'(t) = f(x(t)) for all $t \in I$ and $x(t_0) = x_0$.

Converting non-autonomous systems

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A non-autonomous system x' = g(x, t) can be converted into an autonomous system by letting $x_{n+1} = t$ and $x'_{n+1} = 1$.

${\sf Goals/Questions}$

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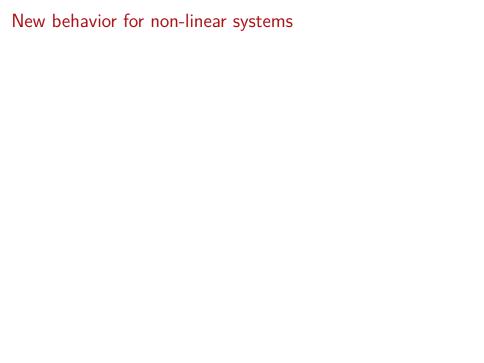
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- ▶ How do solutions change if *f* changes slightly?
- Consider the size of the interval on which the solution exists.



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Solutions may not be defined on all of \mathbb{R} .

Key idea

We have solved the initial value problem x'(t) = f(x(t)) with $x(0) = x_0$ if we can find a continuous function x(t) satisfying

$$x(t) = x_0 + \int_{s=0}^{t} f(x(s)) ds$$

for all $t \in [-a, a]$ for some a > 0.

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What happens when we take limits on both sides of the equation defining u_{k+1} ?

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First convert to an autonomous system via $x_1 = x$ and $x_2 = t$:

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} x_1 x_2 \\ 1 \end{pmatrix} =: f(x_1, x_2)$$

with initial condition
$$\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
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