Math 322

March 2, 2022

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Solution: An interval I containing t_0 and a parametrized curve $x\colon I \to E \subseteq \mathbb{R}^n$ with $x'(t) = f(x(t))$ for all $t \in I$ and $x(t_0) = x_0$.

Converting non-autonomous systems

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$f : E \to \mathbb{R}^n$ is *autonomous*, i.e., f does not depend on t

A non-autonomous system $x' = g(x, t)$ can be converted into an autonomous system by letting $x_{n+1} = t$ and $x'_{n+1} = 1$.

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- Consider the size of the interval on which the solution exists.

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Solutions may not be defined on all of $\mathbb{R}$.

Key idea

We have solved the initial value problem $x'(t) = f(x(t))$ with $x(0) = x_0$ if we can find a continuous function $x(t)$ satisfying

$$x(t) = x_0 + \int_{s=0}^{t} f(x(s))\, ds$$

for all $t \in [-a, a]$ for some $a > 0$.

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What happens when we take limits on both sides of the equation defining u_{k+1}?

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First convert to an autonomous system via $x_1 = x$ and $x_2 = t$:

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} x_1 x_2 \\ 1 \end{pmatrix} =: f(x_1, x_2)$$

with initial condition $\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.