Math 322

February 28, 2022

n-th order linear homogeneous equations revisited

Consider the equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0 \qquad (1)$$

with initial values for $y(0), y'(0), \ldots, y^{(n-1)}$ specified.

n-th order linear homogeneous equations revisited

Consider the equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0 \tag{1}$$

with initial values for $y(0), y'(0), \ldots, y^{(n-1)}$ specified.

char. poly: $P(x) = x^n + a_n x^{n-1} + \cdots + a_0$

n-th order linear homogeneous equations revisited

Consider the equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0 \tag{1}$$

with initial values for $y(0), y'(0), \ldots, y^{(n-1)}$ specified.

char. poly: $P(x) = x^n + a_n x^{n-1} + \cdots + a_0 = \prod_{i=1}^{k}(x - \lambda_i)^{m_i}$.

n-th order linear homogeneous equations revisited

Consider the equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0 \qquad (1)$$

with initial values for $y(0), y'(0), \ldots, y^{(n-1)}$ specified.

char. poly: $P(x) = x^n + a_n x^{n-1} + \cdots + a_0 = \prod_{i=1}^{k}(x - \lambda_i)^{m_i}$.

basic solutions: $\{t^j e^{\lambda_i} : 0 \leq j < m_i, 1 \leq i \leq k\}$.

n-th order linear homogeneous equations revisited

Consider the equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0 \qquad (1)$$

with initial values for $y(0), y'(0), \ldots, y^{(n-1)}$ specified.

char. poly: $P(x) = x^n + a_n x^{n-1} + \cdots + a_0 = \prod_{i=1}^{k}(x - \lambda_i)^{m_i}$.

basic solutions: $\{t^j e^{\lambda_i} : 0 \leq j < m_i, 1 \leq i \leq k\}$.

Claim: The solution to eqn. (1) is unique and is uniquely expressed as a linear combination of the basic solutions.

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

$$x_1 := y,\ x_2 := y',\ x_3 := y'',\ \ldots\ ,\ x_n := y^{(n-1)}$$

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

$$x_1 := y,\ x_2 := y',\ x_3 := y'',\ \ldots\ ,\ x_n := y^{(n-1)}$$

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ \vdots \\ x_{n-1}' \\ x_n' \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \mathbf{0} \\ & & & \ddots & \ddots & \\ \mathbf{0} & & & & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-2} & -a_{n-1} \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix}$$

$$x_0 = x(0) = (y(0), y'(0), \ldots, y^{(n-1)}(0))$$

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

$$x_1 := y,\ x_2 := y',\ x_3 := y'',\ \ldots\ ,\ x_n := y^{(n-1)}$$

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ \vdots \\ x_{n-1}' \\ x_n' \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \mathbf{0} \\ \mathbf{0} & & & \ddots & \ddots & \\ & & & & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-2} & -a_{n-1} \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix}$$

$x_0 = x(0) = (y(0), y'(0), \ldots, y^{(n-1)}(0))$

Unique solution: $x(t) = e^{At}x_0$

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

$$x_1 := y,\ x_2 := y',\ x_3 := y'',\ \ldots\ ,\ x_n := y^{(n-1)}$$

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ \vdots \\ x_{n-1}' \\ x_n' \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & \mathbf{0} & \\ & & 0 & 1 & & \\ \mathbf{0} & & & \ddots & \ddots & \\ & & & & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-2} & -a_{n-1} \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix}$$

$x_0 = x(0) = (y(0), y'(0), \ldots, y^{(n-1)}(0))$

Unique solution: $x(t) = e^{At}x_0$

First component is our solution: $x_1 = y$.

Proposition. $p_A(x) := \det(A - xI_n) = (-1)^{n+1}P(x)$.

Proposition. $p_A(x) := \det(A - xI_n) = (-1)^{n+1}P(x)$.

Proof.

$$A - xI_n = \begin{pmatrix} -x & 1 & & & & \\ & -x & 1 & & & \mathbf{0} \\ & & -x & 1 & & \\ & \mathbf{0} & & \ddots & \ddots & \\ & & & & -x & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-2} & -x - a_{n-1} \end{pmatrix}$$

Proposition. $p_A(x) := \det(A - xI_n) = (-1)^{n+1}P(x)$.

Proof.

$$A - xI_n = \begin{pmatrix} -x & 1 & & & & \\ & -x & 1 & & \mathbf{0} & \\ & & -x & 1 & & \\ \mathbf{0} & & & \ddots & \ddots & \\ & & & & -x & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-2} & -x - a_{n-1} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & & & & \\ 0 & -x & 1 & & \mathbf{0} & \\ 0 & & -x & 1 & & \\ \vdots & \mathbf{0} & & \ddots & \ddots & \\ 0 & & & & -x & 1 \\ -P(x) & -a_1 & -a_2 & \dots & -a_{n-2} & -a_{n-1} - x \end{pmatrix}$$

Proposition. Let λ be an eigenvalue for A. Then the corresponding eigenspace is

$$E_\lambda = \mathrm{Span}\{(1, \lambda, \lambda^2, \ldots, \lambda^{n-1})\}.$$

Proposition. Let λ be an eigenvalue for A. Then the corresponding eigenspace is

$$E_\lambda = \mathrm{Span}\{(1, \lambda, \lambda^2, \ldots, \lambda^{n-1})\}.$$

Corollary. Suppose that A has distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ (over $\mathbb{C}$) with algebraic multiplicities, $m_1, \ldots, m_k$, respectively, so the its characteristic polynomial is

$$p_A(x) = \prod_{i=1}^{k} (\lambda_i - x)^{m_i}.$$

Proposition. Let λ be an eigenvalue for A. Then the corresponding eigenspace is

$$E_\lambda = \mathrm{Span}\{(1, \lambda, \lambda^2, \ldots, \lambda^{n-1})\}.$$

Corollary. Suppose that A has distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ (over $\mathbb{C}$) with algebraic multiplicities, $m_1, \ldots, m_k$, respectively, so the its characteristic polynomial is

$$p_A(x) = \prod_{i=1}^{k} (\lambda_i - x)^{m_i}.$$

Then the Jordan form for A is

$$\begin{pmatrix} J_{m_1}(\lambda_1) & & & \\ & J_{m_2}(\lambda_2) & & \mathbf{0} \\ & & \ddots & \\ \mathbf{0} & & & J_{m_k}(\lambda_k) \end{pmatrix}.$$

Proposition. Every solution to our original *n*-th order equation (with a given initial condition) is a unique linear combination of the *basic* functions

$$\left\{ t^j e^{\lambda_i t} : 0 \le j < m_i, 1 \le i \le k \right\},$$

and each linear combination of these functions is a solution for some initial condition.

Proposition. Every solution to our original n-th order equation (with a given initial condition) is a unique linear combination of the *basic* functions

$$\left\{ t^j e^{\lambda_i t} : 0 \leq j < m_i, 1 \leq i \leq k \right\},$$

and each linear combination of these functions is a solution for some initial condition.

General idea:

$$e^{J_{m_i}(\lambda_i t)} = e^{\lambda_i t} \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \dots & \dots & \frac{t^{m_i-1}}{(m_i-1)!} \\ 0 & 1 & t & \dots & \dots & \frac{t^{k-2}}{(m_i-2)!} \\ 0 & 0 & 1 & \dots & \dots & \frac{t^{k-3}}{(m_i-3)!} \\ & \ddots & & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & 0 & 1 & t \\ 0 & \dots & \dots & \dots & 0 & 1 \end{pmatrix}.$$

Uniqueness

That the basic solutions *are* solutions is HW.

Uniqueness

That the basic solutions *are* solutions is HW.

To show uniqueness, list the basic functions $f_1, \ldots, f_n$, and for each $(\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$, consider the solution

$$s_\alpha(t) = \alpha_1 f_1 + \ldots \alpha_n f_n.$$

Uniqueness

That the basic solutions *are* solutions is HW.

To show uniqueness, list the basic functions $f_1, \ldots, f_n$, and for each $(\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$, consider the solution

$$s_\alpha(t) = \alpha_1 f_1 + \ldots \alpha_n f_n.$$

Define the linear function:

$$\begin{aligned} \phi \colon \mathbb{C}^n &\to \mathbb{C}^n \\ (\alpha_1, \ldots, \alpha_n) &\mapsto (s_\alpha(0), s'_\alpha(0), \ldots, s_\alpha^{(n-1)}(0)). \end{aligned}$$

Uniqueness

That the basic solutions *are* solutions is HW.

To show uniqueness, list the basic functions $f_1, \ldots, f_n$, and for each $(\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$, consider the solution

$$s_\alpha(t) = \alpha_1 f_1 + \ldots \alpha_n f_n.$$

Define the linear function:

$$\begin{aligned} \phi \colon \mathbb{C}^n &\to \mathbb{C}^n \\ (\alpha_1, \ldots, \alpha_n) &\mapsto (s_\alpha(0), s'_\alpha(0), \ldots, s_\alpha^{(n-1)}(0)). \end{aligned}$$

Then ϕ is surjective

Uniqueness

That the basic solutions *are* solutions is HW.

To show uniqueness, list the basic functions $f_1, \ldots, f_n$, and for each $(\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$, consider the solution

$$s_\alpha(t) = \alpha_1 f_1 + \ldots \alpha_n f_n.$$

Define the linear function:

$$\phi \colon \mathbb{C}^n \to \mathbb{C}^n$$
$$(\alpha_1, \ldots, \alpha_n) \mapsto (s_\alpha(0), s'_\alpha(0), \ldots, s_\alpha^{(n-1)}(0)).$$

Then ϕ is surjective, hence injective. Done.