Math 322

March 4, 2022

Banach spaces

Let $(V, \| \ \|)$ be a normed vector space over $F = \mathbb{R}$ or $\mathbb{C}$.

Banach spaces

Let $(V, \| \ \|)$ be a normed vector space over $F = \mathbb{R}$ or $\mathbb{C}$. Recall this means that for all $v, w \in V$ and $\alpha \in F$,

1. $\|v\| \geq 0$ with equality if and only if $v = 0$;
2. $\|\alpha v\| = |\alpha| \|v\|$;
3. $\|v + w\| \leq \|v\| + \|w\|$.

Banach spaces

Let $(V, \| \ \|)$ be a normed vector space over $F = \mathbb{R}$ or $\mathbb{C}$. Recall this means that for all $v, w \in V$ and $\alpha \in F$,

1. $\|v\| \geq 0$ with equality if and only if $v = 0$;
2. $\|\alpha v\| = |\alpha| \|v\|$;
3. $\|v + w\| \leq \|v\| + \|w\|$.

If every Cauchy sequence in V converges (in V), then we say V is *complete*, and in that case $(V, \| \ \|)$ is called a *Banach space*.

Banach spaces

Let $(V, \| \ \|)$ be a normed vector space over $F = \mathbb{R}$ or $\mathbb{C}$. Recall this means that for all $v, w \in V$ and $\alpha \in F$,

1. $\|v\| \geq 0$ with equality if and only if $v = 0$;
2. $\|\alpha v\| = |\alpha|\|v\|$;
3. $\|v + w\| \leq \|v\| + \|w\|$.

If every Cauchy sequence in V converges (in V), then we say V is *complete*, and in that case $(V, \| \ \|)$ is called a *Banach space*.

Definition. A *Banach space* is a complete normed linear space.

Contraction mappings

Definition. Let $(V, \| \ \|)$ be a Banach space, and let $X \subseteq V$.
Let $T\colon X \to X$.

Contraction mappings

Definition. Let $(V, \| \ \|)$ be a Banach space, and let $X \subseteq V$. Let $T\colon X \to X$.

1. A point $u \in X$ is a *fixed point* for T if $T(u) = u$.

Contraction mappings

Definition. Let $(V, \| \ \|)$ be a Banach space, and let $X \subseteq V$. Let $T \colon X \to X$.

1. A point $u \in X$ is a *fixed point* for T if $T(u) = u$.
2. The function T is a *contraction mapping* if there is a constant $c \in [0, 1) \subset \mathbb{R}$ such that

   $$\|T(u) - T(v)\| \leq c\|u - v\|$$

   for all $u, v \in X$.

Contraction mapping principle

Theorem. Let $(V, \| \ \|)$ be a Banach space, and let $X \subseteq V$ be a closed subset of V (hence, it contains all of its limit points).

Contraction mapping principle

Theorem. Let $(V, \| \ \|)$ be a Banach space, and let $X \subseteq V$ be a closed subset of V (hence, it contains all of its limit points).

Suppose that $T \colon X \to X$ is a contraction mapping and fix a constant $c \in [0, 1) \subset \mathbb{R}$ so that

$$\|T(u) - T(v)\| \leq c\|u - v\|$$

for all $u, v \in X$.

Contraction mapping principle

Theorem. Let $(V, \| \ \|)$ be a Banach space, and let $X \subseteq V$ be a closed subset of V (hence, it contains all of its limit points).

Suppose that $T\colon X \to X$ is a contraction mapping and fix a constant $c \in [0, 1) \subset \mathbb{R}$ so that

$$\|T(u) - T(v)\| \leq c\|u - v\|$$

for all $u, v \in X$.

Then T has a unique fixed point $\tilde{u} \in X$.

Contraction mapping principle

Theorem. Let $(V, \| \ \|)$ be a Banach space, and let $X \subseteq V$ be a closed subset of V (hence, it contains all of its limit points).

Suppose that $T\colon X \to X$ is a contraction mapping and fix a constant $c \in [0, 1) \subset \mathbb{R}$ so that

$$\|T(u) - T(v)\| \leq c\|u - v\|$$

for all $u, v \in X$.

Then T has a unique fixed point $\tilde{u} \in X$.

Fix any $u_0 \in X$. Then for all $m \geq 0$,

$$\|\tilde{u} - T^m(u_0)\| \leq \frac{c^m}{1 - c}\|T(u_0) - u_0\|.$$

Contraction mapping principle

Theorem. Let $(V, \| \, \|)$ be a Banach space, and let $X \subseteq V$ be a closed subset of V (hence, it contains all of its limit points).

Suppose that $T\colon X \to X$ is a contraction mapping and fix a constant $c \in [0,1) \subset \mathbb{R}$ so that

$$\|T(u) - T(v)\| \leq c\|u - v\|$$

for all $u, v \in X$.

Then T has a unique fixed point $\tilde{u} \in X$.

Fix any $u_0 \in X$. Then for all $m \geq 0$,

$$\|\tilde{u} - T^m(u_0)\| \leq \frac{c^m}{1-c}\|T(u_0) - u_0\|.$$

In particular, $\lim_{m\to\infty} T^m(u_0) = \tilde{u}$.

Contraction mapping principle

Theorem. Let $(V, \| \ \|)$ be a Banach space, and let $X \subseteq V$ be a closed subset of V (hence, it contains all of its limit points).

Suppose that $T \colon X \to X$ is a contraction mapping and fix a constant $c \in [0, 1) \subset \mathbb{R}$ so that

$$\|T(u) - T(v)\| \leq c\|u - v\|$$

for all $u, v \in X$.

Then T has a unique fixed point $\tilde{u} \in X$.

Fix any $u_0 \in X$. Then for all $m \geq 0$,

$$\|\tilde{u} - T^m(u_0)\| \leq \frac{c^m}{1-c}\|T(u_0) - u_0\|.$$

In particular, $\lim_{m\to\infty} T^m(u_0) = \tilde{u}$. *Proof . . .*

Method of successive approximations

To solve $x'(t) = f(x(t))$ with $x(0) = x_0$,

Method of successive approximations

To solve $x'(t) = f(x(t))$ with $x(0) = x_0$, create the sequence of functions

$$\begin{aligned} u_0 &:= x_0 \\ u_{k+1} &:= x_0 + \int_{s=0}^{t} f(u_k(s))\, ds, \quad \text{for } k \geq 0. \end{aligned}$$

Method of successive approximations

To solve $x'(t) = f(x(t))$ with $x(0) = x_0$, create the sequence of functions

$$\begin{aligned} u_0 &:= x_0 \\ u_{k+1} &:= x_0 + \int_{s=0}^{t} f(u_k(s))\, ds, \quad \text{for } k \geq 0. \end{aligned}$$

Operator on $(C(I), \| \,\|)$:

$$\begin{aligned} T\colon C(I) &\to C(I) \\ u &\mapsto x_0 + \int_{s=0}^{t} f(u(s))\, ds. \end{aligned}$$

Method of successive approximations

Operator on $(C(I), \| \ \|)$:

$$\begin{aligned} T\colon C(I) &\to C(I) \\ u &\mapsto x_0 + \int_{s=0}^{t} f(u(s))\, ds. \end{aligned}$$

Method of successive approximations

Operator on $(C(I), \| \ \|)$:

$$\begin{aligned} T\colon C(I) &\to C(I) \\ u &\mapsto x_0 + \int_{s=0}^{t} f(u(s))\, ds. \end{aligned}$$

Definition. If $I \subset \mathbb{R}$ is a closed bounded interval, let $C(I)$ denote the $\mathbb{R}$-vector space of continuous functions on $I \to \mathbb{R}^n$ (where n is fixed). For each $u \in C(I)$, define

$$\|u\| := \sup_{t \in I} |u(t)| = \max_{t \in I} |u(t)|.$$

Method of successive approximations

Operator on $(C(I), \| \ \|)$:

$$T\colon C(I) \to C(I)$$
$$u \mapsto x_0 + \int_{s=0}^{t} f(u(s))\, ds.$$

Definition. If $I \subset \mathbb{R}$ is a closed bounded interval, let $C(I)$ denote the $\mathbb{R}$-vector space of continuous functions on $I \to \mathbb{R}^n$ (where n is fixed). For each $u \in C(I)$, define

$$\|u\| := \sup_{t \in I} |u(t)| = \max_{t \in I} |u(t)|.$$

Under what conditions is T a contraction mapping?