Math 322

February 23, 2022

HW4, Problem 3

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To prove this, fix t and define the sequence

$$x_n = \left(\sum_{k=0}^{n} \frac{A^k t^k}{k!} \right) x_0$$

for each $n \geq 0$. Since $Ax_0 \in W$, it easily follows that $x_n \in W$ for all n.

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Since W is complete, it suffices to show that (x_n) is Cauchy.

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$$\left\| \sum_{k=0}^{n} \frac{A^k t^k}{k!} - \sum_{k=0}^{m} \frac{A^k t^k}{k!} \right\| = \left\| \sum_{k=m+1}^{n} \frac{A^k t^k}{k!} \right\| < \varepsilon / |x_0|.$$

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It then follows that

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□

Trig sum formulas review

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$$= (\cos(\theta) + i\sin(\theta))(\cos(\psi) + i\sin(\psi))$$

Trig sum formulas review

$$\begin{aligned}
\cos(\theta + \psi) + i\ \sin(\theta + \psi) &= e^{i(\theta+\psi)} = e^{i\theta} e^{i\psi} \\
&= (\cos(\theta) + i \sin(\theta))(\cos(\psi) + i \sin(\psi)) \\
&= (\cos(\theta)\cos(\psi) - \sin(\theta)\sin(\psi)) \\
&\qquad + i(\cos(\theta)\sin(\psi) + \cos(\psi)\sin(\psi))
\end{aligned}$$

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Chebyshev polynomials

Nonhomogeneous linear systems

Proposition. Let $A \in M_n(F)$ and consider the system

$$x'(t) = Ax(t) + b(t)$$

where $t \mapsto b(t) \in F^n$ is continuous. The solution with initial condition x_0 is

$$x(t) = e^{At}x_0 + e^{At}\int_{s=0}^{t} e^{-As}b(s)\,ds.$$

The solution is unique.

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Apply to forced harmonic oscillator:

$$x'' = -x + f(t).$$

Forced harmonic oscillator

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Solution:

$$x(t) = x(0)\cos(t) + x'(0)\sin(t) + \int_{s=0}^{t} f(s)\sin(t-s)\,ds.$$

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Suppose $f(t) = \cos(\omega t)$.

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Suppose $f(t) = \cos(\omega t)$. Then

$$x(t) = x(0)\cos(t) + x'(0)\sin(t) + \frac{\cos(\omega t) - \cos(t)}{1 - \omega^2}.$$

$$x'' = -x + \cos(\omega t)$$

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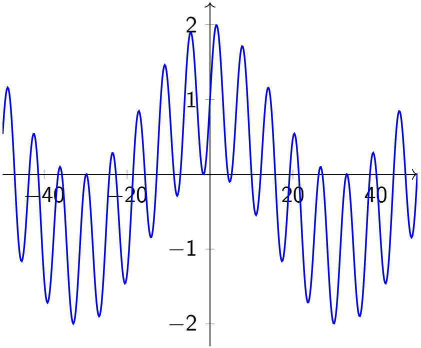

$x(0) = x'(0) = 1$, $\omega = 0.1$