Math 322

February 21, 2022

Outline

- Stability: stable, center, and unstable spaces

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- Linear systems in $\mathbb{R}^3$

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- Linear systems in $\mathbb{R}^3$
- Inhomogeneous systems

Stable, center, and unstable spaces for a linear system

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stable, *center*, and *unstable* subspaces for A:

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Proposition. Each generalized eigenspace, the stable, center, and unstable spaces are *invariant* under A and under e^{At} for all $t \in \mathbb{R}$.

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I. $u, v, w \in \mathbb{R}$:

$$J = \begin{pmatrix} u & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & w \end{pmatrix} \qquad x(t) = e^{Jt}x_0 = \begin{pmatrix} e^{ut} & 0 & 0 \\ 0 & e^{vt} & 0 \\ 0 & 0 & e^{wt} \end{pmatrix} x_0.$$

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III. $u \in \mathbb{R}$:

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IV. $a, b, u \in \mathbb{R}$ and $b \neq 0$:

$$J = \begin{pmatrix} a & -b & 0 \\ b & a & 0 \\ 0 & 0 & u \end{pmatrix} \qquad x(t) = e^{Jt}x_0 = \begin{pmatrix} e^{at}\cos(bt) & -e^{at}\sin(bt) & 0 \\ e^{at}\sin(bt) & e^{at}\cos(bt) & 0 \\ 0 & 0 & e^{ut} \end{pmatrix} x_0.$$

Non-homogeneous

Proposition. Let $A \in M_n(F)$ and consider the system

$$x'(t) = Ax(t) + b(t)$$

where $t \mapsto b(t) \in F^n$ is continuous.

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Proposition. Let $A \in M_n(F)$ and consider the system

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where $t \mapsto b(t) \in F^n$ is continuous. The solution with initial condition x_0 is

$$x(t) = e^{At}x_0 + e^{At}\int_{s=0}^{t} e^{-As}b(s)\,ds.$$

The solution is unique.