Math 322

February 16, 2022

Homework 3, Problem 1.5

$$y'' - 6yy' = 0, \quad y(0) = 2,\ y'(0) = 9.$$

Homework 3, Problem 1.5

$y'' - 6yy' = 0, \quad y(0) = 2, \; y'(0) = 9.$

SOLUTION: Substitute $v = y'$. Then

$$y'' = v' = \frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = v\frac{dv}{dy}.$$

Homework 3, Problem 1.5

$y'' - 6yy' = 0, \quad y(0) = 2, \; y'(0) = 9.$

SOLUTION: Substitute $v = y'$. Then

$$y'' = v' = \frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = v\frac{dv}{dy}.$$

Our equation becomes

$$v\frac{dv}{dy} - 6yv = 0,$$

Homework 3, Problem 1.5

$$y'' - 6yy' = 0, \quad y(0) = 2,\ y'(0) = 9.$$

SOLUTION: Substitute $v = y'$. Then

$$y'' = v' = \frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = v\frac{dv}{dy}.$$

Our equation becomes

$$v\frac{dv}{dy} - 6yv = 0,$$

Since we want $v(0) = y'(0) = 1 > 0$, assume $v > 0$ and divide by v:

$$\frac{dv}{dy} - 6y = 0,$$

Homework 3, Problem 1.5

$y'' - 6yy' = 0, \quad y(0) = 2,\ y'(0) = 9.$

SOLUTION: Substitute $v = y'$. Then

$$y'' = v' = \frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = v\frac{dv}{dy}.$$

Our equation becomes

$$v\frac{dv}{dy} - 6yv = 0,$$

Since we want $v(0) = y'(0) = 1 > 0$, assume $v > 0$ and divide by v:

$$\frac{dv}{dy} - 6y = 0,$$

which is separable:

$$\int dv = \int 6y\,dy \quad \Rightarrow \quad v = 3y^2 + c.$$

Homework 3, Problem 1.5

$$v = 3y^2 + c$$

Homework 3, Problem 1.5

$v = 3y^2 + c$

Using the initial conditions $y(0) = 2$ and $v(0) = y'(0) = 9$, we find $c = -3$. Therefore,

$$y' = v = 3y^2 - 3.$$

Homework 3, Problem 1.5

$v = 3y^2 + c$

Using the initial conditions $y(0) = 2$ and $v(0) = y'(0) = 9$, we find $c = -3$. Therefore,

$$y' = v = 3y^2 - 3.$$

This is a separable equation

$$\frac{1}{3}\int \frac{dy}{y^2-1} = \int dt \quad \Rightarrow \quad \frac{1}{6}\int \left(\frac{1}{y-1} - \frac{1}{y+1}\right) = t + a$$

$$\Rightarrow \quad (\ln(y-1) - \ln(y+1)) = 6t + b$$

$$\Rightarrow \quad \frac{y-1}{y+1} = \alpha e^{6t}.$$

Homework 3, Problem 1.5

$v = 3y^2 + c$

Using the initial conditions $y(0) = 2$ and $v(0) = y'(0) = 9$, we find $c = -3$. Therefore,

$$y' = v = 3y^2 - 3.$$

This is a separable equation

$$\frac{1}{3}\int \frac{dy}{y^2-1} = \int dt \quad \Rightarrow \quad \frac{1}{6}\int \left(\frac{1}{y-1} - \frac{1}{y+1}\right) = t + a$$

$$\Rightarrow \quad (\ln(y-1) - \ln(y+1)) = 6t + b$$

$$\Rightarrow \quad \frac{y-1}{y+1} = \alpha e^{6t}.$$

The initial condition $y(0) = 2$ then says $\alpha = 1/3$. So

$$\frac{y-1}{y+1} = \frac{1}{3}e^{6t} \quad \Rightarrow \quad \boxed{y = \frac{3+e^{6t}}{3-e^{6t}}}.$$

Homework 3, Problem 2

$A \in M_n(F)$ with i-th row r_i.

Homework 3, Problem 2

$A \in M_n(F)$ with i-th row r_i.

$\ell = \max\{|r_i| : 1 \leq i \leq n\}$

Homework 3, Problem 2

$A \in M_n(F)$ with i-th row r_i.

$\ell = \max\{|r_i| : 1 \leq i \leq n\}$

Prove $\|A\| \leq \ell\sqrt{n}$.

Homework 3, Problem 2

$A \in M_n(F)$ with i-th row r_i.

$\ell = \max\{|r_i| : 1 \leq i \leq n\}$

Prove $\|A\| \leq \ell\sqrt{n}$.

SOLUTION: Let $x \in F^n$ with $|x| \leq 1$.

Homework 3, Problem 2

$A \in M_n(F)$ with i-th row r_i.

$\ell = \max\{|r_i| : 1 \leq i \leq n\}$

Prove $\|A\| \leq \ell\sqrt{n}$.

Solution: Let $x \in F^n$ with $|x| \leq 1$. By Cauchy-Schwarz,

$$|Ax|^2 = \sum_{i=1}^{n} |r_i \cdot x|^2$$

Homework 3, Problem 2

$A \in M_n(F)$ with i-th row r_i.

$\ell = \max\{|r_i| : 1 \leq i \leq n\}$

Prove $\|A\| \leq \ell\sqrt{n}$.

Solution: Let $x \in F^n$ with $|x| \leq 1$. By Cauchy-Schwarz,

$$|Ax|^2 = \sum_{i=1}^{n} |r_i \cdot x|^2 \leq \sum_{i=1}^{n} |r_i|^2 |x|^2$$

Homework 3, Problem 2

$A \in M_n(F)$ with i-th row r_i.

$\ell = \max\{|r_i| : 1 \leq i \leq n\}$

Prove $\|A\| \leq \ell\sqrt{n}$.

Solution: Let $x \in F^n$ with $|x| \leq 1$. By Cauchy-Schwarz,

$$|Ax|^2 = \sum_{i=1}^{n} |r_i \cdot x|^2 \leq \sum_{i=1}^{n} |r_i|^2 |x|^2 \leq \sum_{i=1}^{n} |r_i|^2$$

Homework 3, Problem 2

$A \in M_n(F)$ with i-th row r_i.

$\ell = \max\{|r_i| : 1 \leq i \leq n\}$

Prove $\|A\| \leq \ell\sqrt{n}$.

SOLUTION: Let $x \in F^n$ with $|x| \leq 1$. By Cauchy-Schwarz,

$$|Ax|^2 = \sum_{i=1}^{n} |r_i \cdot x|^2 \leq \sum_{i=1}^{n} |r_i|^2 |x|^2 \leq \sum_{i=1}^{n} |r_i|^2 \leq \sum_{i=1}^{n} \ell^2$$

Homework 3, Problem 2

$A \in M_n(F)$ with i-th row r_i.

$\ell = \max\{|r_i| : 1 \leq i \leq n\}$

Prove $\|A\| \leq \ell\sqrt{n}$.

Solution: Let $x \in F^n$ with $|x| \leq 1$. By Cauchy-Schwarz,

$$|Ax|^2 = \sum_{i=1}^{n} |r_i \cdot x|^2 \leq \sum_{i=1}^{n} |r_i|^2 |x|^2 \leq \sum_{i=1}^{n} |r_i|^2 \leq \sum_{i=1}^{n} \ell^2 = \ell^2 n.$$

Homework 3, Problem 2

$A \in M_n(F)$ with i-th row r_i.

$\ell = \max\{|r_i| : 1 \leq i \leq n\}$

Prove $\|A\| \leq \ell\sqrt{n}$.

SOLUTION: Let $x \in F^n$ with $|x| \leq 1$. By Cauchy-Schwarz,

$$|Ax|^2 = \sum_{i=1}^{n} |r_i \cdot x|^2 \leq \sum_{i=1}^{n} |r_i|^2|x|^2 \leq \sum_{i=1}^{n} |r_i|^2 \leq \sum_{i=1}^{n} \ell^2 = \ell^2 n.$$

Therefore, $|Ax| \leq \ell\sqrt{n}$ for all $|x| \leq 1$.

Homework 3, Problem 2

$A \in M_n(F)$ with i-th row r_i.

$\ell = \max\{|r_i| : 1 \leq i \leq n\}$

Prove $\|A\| \leq \ell\sqrt{n}$.

SOLUTION: Let $x \in F^n$ with $|x| \leq 1$. By Cauchy-Schwarz,

$$|Ax|^2 = \sum_{i=1}^{n} |r_i \cdot x|^2 \leq \sum_{i=1}^{n} |r_i|^2|x|^2 \leq \sum_{i=1}^{n} |r_i|^2 \leq \sum_{i=1}^{n} \ell^2 = \ell^2 n.$$

Therefore, $|Ax| \leq \ell\sqrt{n}$ for all $|x| \leq 1$. It follows that

$$\|A\| = \max_{|x|\leq 1} |Ax| \leq \ell\sqrt{n}.$$

Outline

1. Review of diagonalization
2. Jordan form

Jordan matrix

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2+3i \end{pmatrix}$$

Jordan form

Theorem. Let $A \in M_n(\mathbb{C})$.

Jordan form

Theorem. Let $A \in M_n(\mathbb{C})$. Then there exists an invertible matrix $P \in M_n(\mathbb{C})$ such that $P^{-1}AP = J$ where J is a Jordan matrix.

Jordan form

Theorem. Let $A \in M_n(\mathbb{C})$. Then there exists an invertible matrix $P \in M_n(\mathbb{C})$ such that $P^{-1}AP = J$ where J is a Jordan matrix. The matrix J is called the *Jordan form for A*.

Jordan form

Theorem. Let $A \in M_n(\mathbb{C})$. Then there exists an invertible matrix $P \in M_n(\mathbb{C})$ such that $P^{-1}AP = J$ where J is a Jordan matrix. The matrix J is called the *Jordan form for A*. It is unique up to a permutation of the Jordan blocks.

Jordan form

Theorem. Let $A \in M_n(\mathbb{C})$. Then there exists an invertible matrix $P \in M_n(\mathbb{C})$ such that $P^{-1}AP = J$ where J is a Jordan matrix. The matrix J is called the *Jordan form for A*. It is unique up to a permutation of the Jordan blocks.

The diagonal entries of J are exactly the eigenvalues of A repeated according to their algebraic multiplicities.

Jordan form

Theorem. Let $A \in M_n(\mathbb{C})$. Then there exists an invertible matrix $P \in M_n(\mathbb{C})$ such that $P^{-1}AP = J$ where J is a Jordan matrix. The matrix J is called the *Jordan form for A*. It is unique up to a permutation of the Jordan blocks.

The diagonal entries of J are exactly the eigenvalues of A repeated according to their algebraic multiplicities.

The number of blocks having a particular eigenvalue λ along the diagonal is the geometric multiplicity of λ (i.e., $\dim(A - \lambda I_n)$).

Real Jordan form

$$\begin{pmatrix} \lambda & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{\lambda} & 1 & 0 \\ 0 & 0 & 0 & 0 & \bar{\lambda} & 1 \\ 0 & 0 & 0 & 0 & 0 & \bar{\lambda} \end{pmatrix}$$

Real Jordan form

$$\begin{pmatrix} \lambda & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{\lambda} & 1 & 0 \\ 0 & 0 & 0 & 0 & \bar{\lambda} & 1 \\ 0 & 0 & 0 & 0 & 0 & \bar{\lambda} \end{pmatrix} \quad \rightsquigarrow \quad \begin{pmatrix} a & -b & 1 & 0 & 0 & 0 \\ b & a & 0 & 1 & 0 & 0 \\ 0 & 0 & a & -b & 1 & 0 \\ 0 & 0 & b & a & 0 & 1 \\ 0 & 0 & 0 & 0 & a & -b \\ 0 & 0 & 0 & 0 & b & a \end{pmatrix}$$

Real Jordan form

If $A \in M_n(\mathbb{R})$, there exists an invertible $P \in M_n(\mathbb{R})$ such that $P^{-1}AP = J$ where J consists of Jordan blocks—the usual ones for real eigenvalues, and these modified block matrices for conjugate pairs of complex eigenvalues.

Real Jordan form

If $A \in M_n(\mathbb{R})$, there exists an invertible $P \in M_n(\mathbb{R})$ such that $P^{-1}AP = J$ where J consists of Jordan blocks—the usual ones for real eigenvalues, and these modified block matrices for conjugate pairs of complex eigenvalues. The form is unique up to permutation of the blocks and swaps

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \quad \longleftrightarrow \quad \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

Real Jordan form

$$\begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 3 \end{pmatrix}$$