Math 322

February 14, 2022

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3. A has a pair of conjugate complex roots $a \pm bi$ with $b \neq 0$:

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Determinant and trace

Lemma. Let $A \in M_n(F)$ with eigenvalues $\lambda_1, \ldots, \lambda_n$. Then

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(ii) Consider the characteristic polynomial of A:

$$p(x) = \det(A - xI_n).$$

Then the coefficient of x^{n-1} in $p(x)$ is $(-1)^{n-1}\mathrm{trace}(A)$ and the constant term of $p(x)$ is $\det(A)$.

Moduli space for systems in $\mathbb{R}^2$