# Math 322

February 18, 2022

# Exponentiation of a Jordan matrix

$$\left(\begin{array}{c|ccc|cc|cccc}
4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 3 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 3 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 3 & -2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 3 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 3
\end{array}\right)$$

# Exponentiation of a Jordan matrix

$$J := \begin{pmatrix} J_{k_1}(\lambda_1) & & & & \\ & J_{k_2}(\lambda_2) & & \text{\Large 0} & \\ & & J_{k_3}(\lambda_3) & & \\ & \text{\Large 0} & & \ddots & \\ & & & & J_{k_\ell}(\lambda_\ell) \end{pmatrix}$$

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$$e^{Jt} := \begin{pmatrix} e^{J_{k_1}(\lambda_1)t} & & & & \\ & e^{J_{k_2}(\lambda_2)t} & & \mathbf{0} & \\ & & e^{J_{k_3}(\lambda_3)t} & & \\ & \mathbf{0} & & \ddots & \\ & & & & e^{J_{k_\ell}(\lambda_\ell t)} \end{pmatrix}$$

# Exponentiation of a Jordan block

$$e^{J_k(\lambda)t} = e^{(\lambda I_k + N_k)t} = e^{\lambda t I_k} e^{N_k t}$$

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$$\begin{aligned} e^{J_k(\lambda)t} &= e^{(\lambda I_k + N_k)t} = e^{\lambda t I_k} e^{N_k t} \\ &= e^{\lambda t} \left( I_k + t N_k + \frac{t^2}{2} N_k^2 + \frac{t^3}{3!} N_k^3 + \cdots + \frac{t^{k-1}}{(k-1)!} N_k^{k-1} \right) \end{aligned}$$

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&= e^{\lambda t} \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \dots & \dots & \frac{t^{k-1}}{(k-1)!} \\ 0 & 1 & t & \dots & \dots & \frac{t^{k-2}}{(k-2)!} \\ 0 & 0 & 1 & \dots & \dots & \frac{t^{k-3}}{(k-3)!} \\ & \ddots & & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & 0 & 1 & t \\ 0 & \dots & \dots & \dots & 0 & 1 \end{pmatrix}
\end{aligned}
$$

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$$e^{Jt} = e^{at} \begin{pmatrix} R & tR & \frac{t^2}{2!}R & \dots & \dots & \frac{t^{k-1}}{(k-1)!}R \\ 0 & R & tR & \dots & \dots & \frac{t^{k-2}}{(k-2)!}R \\ 0 & 0 & R & \dots & \dots & \frac{t^{k-3}}{(k-3)!}R \\ & \ddots & & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & 0 & R & tR \\ 0 & \dots & \dots & \dots & 0 & R \end{pmatrix}$$

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$$\delta_\ell := \delta_\ell(\lambda) := \dim \ker(A - \lambda I)^\ell$$

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$$\delta_\ell = \begin{cases} \ell & \text{for } 0 \leq \ell \leq k, \\ k & \text{for } \ell > k. \end{cases}$$

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$$\#\ k \times k \text{ Jordan blocks for } \lambda = \begin{cases} 2\delta_1 - \delta_2 & \text{for } k = 1, \\ 2\delta_k - \delta_{k+1} - \delta_{k-1} & \text{for } 1 < k < n, \\ \delta_n - \delta_{n-1} & \text{for } k = n. \end{cases}$$

# Algorithm for computing Jordan form

$$\ker(A - \lambda I) \subseteq \ker(A - \lambda I)^2 \subseteq \ker(A - \lambda I)^3 \subseteq \dots$$

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$$\begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$$

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$$Ae_1 = \lambda e_1,\ Ae_2 = e_1 + \lambda e_2,\ Ae_3 = e_2 + \lambda e_3,\ Ae_4 = e_3 + \lambda e_4.$$

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$$\begin{aligned} (A - \lambda I)e_1 &= 0 \\ (A - \lambda I)e_2 &= e_1 \\ (A - \lambda I)e_3 &= e_2 \\ (A - \lambda I)e_4 &= e_3, \end{aligned}$$