# Math 322

February 9, 2022

# Announcements

- Job talks
- Status of the Stats program: today at 4:10 pm in Lib 389
- Questions?

HW 2, problem 2: $2ty\,y' = t^2 + y^2, \quad y(1) = 0$

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Integrate using the fact that $t \approx 1$ and $v \approx 0$:

$$-\ln(1 - v^2) = \ln(t) + c \quad \Rightarrow \quad 1 - v^2 = \frac{a}{t} \quad \Rightarrow \quad 1 - \frac{y^2}{t^2} = \frac{a}{t}$$

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Use initial condition and solve for $y^2$: $\quad y^2 = t^2 - t = t(t-1)$

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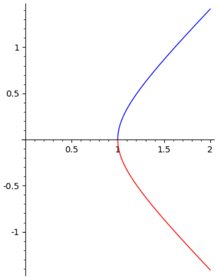
Two possible solutions for $t \geq 1$: $y = \pm\sqrt{t^2 - t}$.



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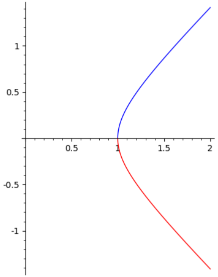


What is the speed of each solution when $t = 1$?

# The Fundamental Theorem for Linear Systems (p. 17)

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Let $A \in M_n(F)$, and let $x_0 \in F^n$. The initial value problem

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**Theorem.** For all $A \in M_n(F)$ and $t_0 > 0$, the function $\mathbb{R} \to M_n(F)$ given by

$$t \mapsto \sum_{k \geq 0} \frac{A^k t^k}{k!} =: e^{At}$$

converges absolutely and uniformly for $t \in [-t_0, t_0]$.

# Cauchy sequences

**Definition.** A sequence $(v_k)$ in a normed vector space $(V, \| \ \|)$ is a *Cauchy sequence* if for all $\varepsilon > 0$ there exists $N \in \mathbb{R}$ such that for all $m, n > N$, we have

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Easy result: every convergent sequence is Cauchy.

Theorem from analysis: if $V$ is a finite-dimensional normed vector space, then $V$ is *complete*: a sequence in $V$ converges if and only if it is Cauchy.

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$$\|f_k(x)\| \leq M_k$$

for all $x \in C$ and for all $k$.

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**Proof.** On board.

# Convergence of exponential function

**Theorem.** For all $A \in M_n(F)$ and $t_0 > 0$, the function $\mathbb{R} \to M_n(F)$ given by

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4. $e^{-A} = \left(e^A\right)^{-1}$.

# Example

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

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Show that $e^{A+B} \neq e^A e^B$. (Note that $AB \neq BA$.)