Math 322

February 7, 2022

Announcements

- Job talks

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- questions?

The Fundamental Theorem for Linear Systems (p. 17)

Let $F = \mathbb{R}$ or $\mathbb{C}$.

Theorem. Let $A \in M_n(F)$, and let $x_0 \in F^n$. The initial value problem

$$\begin{aligned} x' &= Ax \\ x(0) &= x_0 \end{aligned}$$

has the unique solution

$$x = e^{At} x_0.$$

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2. (absolute homogeneity) $\|\alpha v\| = |\alpha|\|v\|$ for all $v \in V$ and $\alpha \in F$.
3. (triangle inequality) $\|v + w\| \leq \|v\| + \|w\|$ for all $v, w \in V$.

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2. (symmetry) $d(v, w) = d(w, v)$ for all $v, w \in V$.
3. (triangle inequality) $d(u, w) \leq d(u, v) + d(v, w)$ for all $u, v, w \in V$.

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Proposition. Let $\| \|_1$ and $\| \|_2$ be two norms on a finite-dimensional vector space V over F. Then these norms are *equivalent* in the following sense: there exist positive real numbers a, b such that

$$a\|v\|_2 \leq \|v\|_1 \leq b\|v\|_2$$

for all $v \in V$.

Operator norm

Definition. The *operator norm* on the vector space $M_n(F)$ of $n \times n$ matrices with coefficients in F is given by

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Lemma 1. For all $A, B \in M_n(F)$ and $x \in F^n$,

1. $|Ax| \le \|A\| |x|$.
2. $\|AB\| \le \|A\| \|B\|$.
3. $\|A^k\| \le \|A\|^k$.

The exponential function

Goal for next time:

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Theorem. For all $A \in M_n(F)$ and $t_0 > 0$, the function $\mathbb{R} \to M_n(F)$ given by

$$t \mapsto \sum_{k \geq 0} \frac{A^k t^k}{k!}$$

converges absolutely and uniformly for $t \in [-t_0, t_0]$.