Math 322

February 2, 2022

Announcements

- job talks

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- mathematical writing

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- mathematical writing
- questions?

Bernoulli-type equations revisited

Imagine a moving particle with velocity v and a force $F = F(v)$ acting on the particle against its direction of motion—a frictional force.

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Suppose F has a power series expansion

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Bernoulli-type! What is the behavior?

Linear, homogeneous, constant coefficients, continued

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So r works if and only if $P(r) = 0$.

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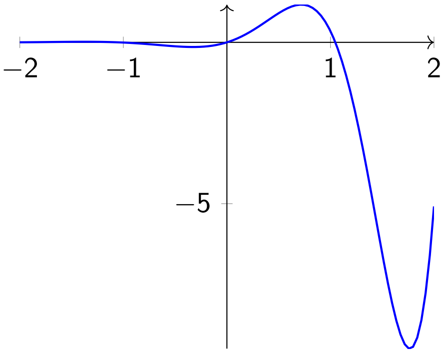

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Then the general solution will have a summand of the form

$$a_0 e^{\lambda t} + a_1 t e^{\lambda t} + \cdots + a_k t^{k-1} e^{\lambda t}.$$

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  Solution:

  $$\begin{aligned} y &= a_1 + a_2t + a_3t^2 + b_1e^{2t} + b_2te^{2t} \\ &\quad + c_1\cos(3t) + c_2\sin(3t) + c_3t\cos(3t) + c_4t\sin(3t) \end{aligned}$$

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where y_p is a particular solution, and y_h is the general solution to the corresponding *homogeneous* equation $P(D)y = 0$.

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$f(t)$	guess for form of y_p
polynomial	general polynomial of some degree
e^{rt}	ae^{rt}
(poly)e^{rt}	(general poly)e^{rt}
$\cos(\omega t)$ or $\sin(\omega t)$	$a\cos(\omega t) + b\sin(\omega t)$
(poly)$e^{rt}\cos(\omega t)$ or (poly)$e^{rt}\sin(\omega t)$	(gen poly)$e^{rt}\cos(\omega t)$ + (gen poly)$e^{rt}\sin(\omega t)$

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$$y'' - 2y' + y = t^2$$

$$y = -5e^t - te^t + 6 + 4t + t^2$$

Graph of solution for $y(0) = 1$, $y'(0) = 2$: