Math 322

January 24, 2022

I. Separable equations

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It is solved by integration:

$$\int p(y)\,dy = \int q(t)\,dt.$$

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General implicit solution:

$$y(t)^2 = 3t^2 + c$$

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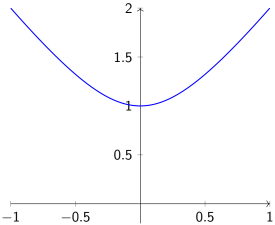
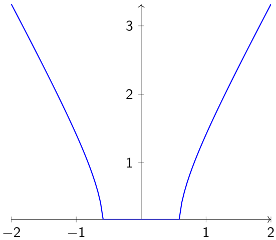


$$y(t) = \sqrt{3t^2 + 1}$$



$$y(t) = \sqrt{3t^2 - 1}$$

Exponential growth and decay

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Solution:

$$y(t) = y_0 e^{rt}$$

Exponential growth and decay

Example. If $y(t) = ae^{rt}$ with $y(0) = a \neq 0$ at what time t has the population doubled?

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Note that $y(t) \to S$ as $t \to \infty$.

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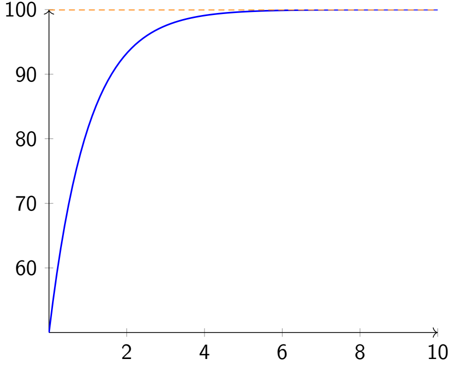

$S = 100$, $I = 50$, and $r = 1$