Math 322

January 28, 2022

Announcements

- Solutions to Wednesday's practice problems.

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- Link from Olly.

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determines y (up to constant).

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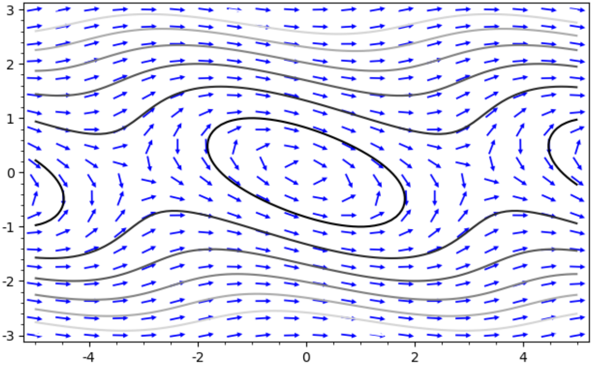
Example.

$$\sin(t+y) + (2y + \sin(t+y))y' = 0$$

Example

Slope field and solutions to

$$\sin(t+y) + (2y + \sin(t+y))y' = 0$$



Relation to potential functions and gradient vector fields

Consider the vector field:

$$\begin{aligned} F\colon \mathbb{R}^2 &\to \mathbb{R}^2 \\ (t, y) &\mapsto (M(t, y), N(t, y)) \end{aligned}$$

Relation to potential functions and gradient vector fields

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Sage demonstration.

Integrating factor to force exactness

If

$$M(t, y) + N(t, y)\,\frac{dy}{dt}$$

is not exact, look for function $\mu(t, y)$ so that

$$\mu(t, y)M(t, y) + \mu(t, y)N(t, y)\,\frac{dy}{dt} = 0,$$

is exact.

Integrating factor example

$$ty^2 + 4t^2y + (3t^2y + 4t^3)\frac{dy}{dt} = 0$$

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Exactness:

$$\frac{\partial}{\partial y}(t^{m+1}y^{n+2} + 4t^{m+2}y^{n+1}) = \frac{\partial}{\partial t}(3t^{m+2}y^{n+1} + 4t^{m+3}y^n).$$

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$$(n+2)t^{m+1}y^{n+1}+4(n+1)t^{m+2}y^n = 3(m+2)t^{m+1}y^{n+1}+4(m+3)t^{m+2}y^n$$

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$$(n+2)t^{m+1}y^{n+1}+4(n+1)t^{m+2}y^n = 3(m+2)t^{m+1}y^{n+1}+4(m+3)t^{m+2}y^n$$

$$m = -1 \quad \text{and} \quad n = 1$$

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Solution:

$$ty^3 + 2t^2y^2 = c$$

Integrating factors

Integrating factors always exist!

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Integrating factors always exist! Although the may be hard to find. See the lecture notes.