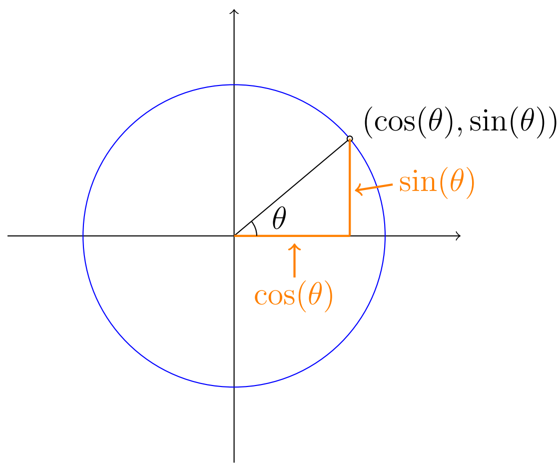
Definition. The *cosine* and *sine* are functions of a real number θ so that the coordinates of the point on the unit circle centered at the origin determined by the angle θ has $(\cos(\theta), \sin(\theta))$ as in the following picture:



The tangent, cotangent, secant, and cosecant functions are defined in terms of cosine and sine:

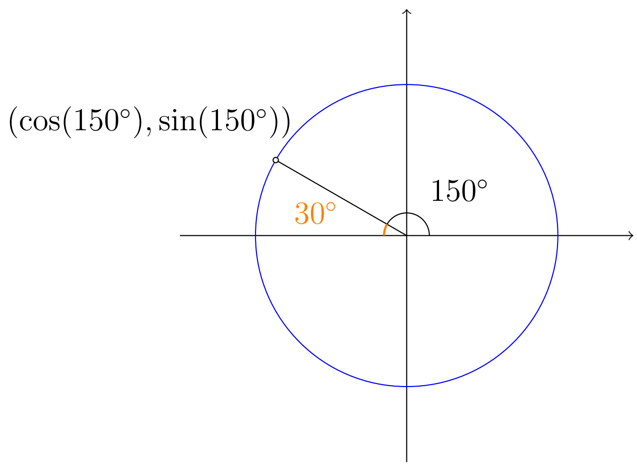
$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}, \quad \cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}, \quad \sec(\theta) = \frac{1}{\cos(\theta)}, \quad \csc(\theta) = \frac{1}{\sin(\theta)},$$

Standard angles. The values of cosine and sine at the standard angles:

θ	$\cos(\theta)$	$\sin(\theta)$
0	1	0
$30° = \frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$45° = \frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$60° = \frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$90° = \frac{\pi}{2}$	0	1

From these angles in the first quadrant, you can derive values for cosine and sine for corresponding angles in the other quadrants.

Examples. The angle 150° is 30° up from 180° as shown below:



Therefore,

$$(\cos(150°), \sin(150°)) = (-\cos(30°), \sin(30)) = \left(-\tfrac{\sqrt{3}}{2}, \tfrac{1}{2}\right).$$

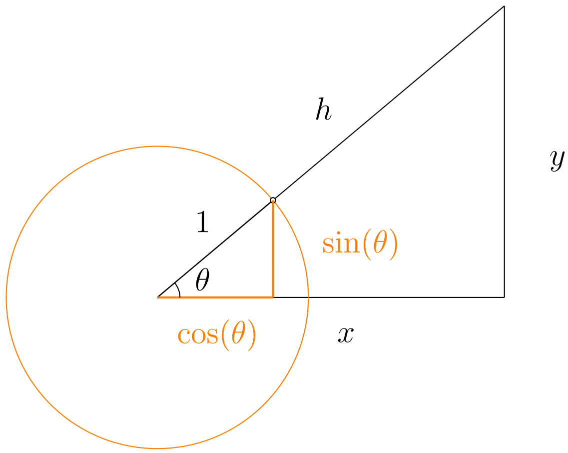
Converting between radians and degrees. If

$$\theta = x \text{ radians} = y \text{ degrees},$$

then "x is to π as y is to 180":

$$\frac{x}{\pi} = \frac{y}{180}.$$

Similar triangles.



$$\frac{\text{adj}}{\text{hyp}} = \frac{x}{h} = \frac{\cos(\theta)}{1} = \cos(\theta)$$

$$\frac{\text{opp}}{\text{hyp}} = \frac{y}{h} = \frac{\sin(\theta)}{1} = \sin(\theta)$$

$$\frac{\text{opp}}{\text{adj}} = \frac{y}{x} = \frac{\sin(\theta)}{\cos(\theta)} = \tan(\theta)$$

$$\frac{\text{adj}}{\text{opp}} = \frac{x}{y} = \frac{\cos(\theta)}{\sin(\theta)} = \cot(\theta)$$

$$\frac{\text{hyp}}{\text{adj}} = \frac{h}{x} = \frac{1}{\cos(\theta)} = \sec(\theta)$$

$$\frac{\text{hyp}}{\text{opp}} = \frac{h}{y} = \frac{1}{\sin(\theta)} = \csc(\theta)$$

Identities. Since $(\cos(\theta), \sin(\theta))$ is a point on the unit circle,

$$\cos^2(\theta) + \sin^2(\theta) = 1.$$

The **sum formulas**:

$$\begin{aligned} \cos(\theta + \psi) &= \cos(\theta)\cos(\psi) - \sin(\theta)\sin(\psi) \\ \sin(\theta + \psi) &= \sin(\theta)\cos(\psi) + \sin(\psi)\cos(\theta). \end{aligned}$$

Letting $\theta = \psi$ in the sum formulas gives the **double-angle** formulas:

$$\begin{aligned} \cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta) \\ \sin(2\theta) &= 2\sin(\theta)\cos(\theta). \end{aligned}$$

Several identities follow from the double-angle formula for cosine. First substitute $\cos^2(\theta) = 1 - \sin^2(\theta)$ to get

$$\cos(2\theta) = 1 - 2\sin^2(\theta).$$

Or, substitute $\sin^2(\theta) = 1 - \cos^2(\theta)$ to get

$$\cos(2\theta) = 2\cos^2(\theta) - 1.$$

Solve for $\sin^2(\theta)$ and for $\cos^2(\theta)$ in these last two equations, respectively, to get the **half-angle** formulas:

$$\begin{aligned} \sin^2(\theta) &= \frac{1 - \cos(2\theta)}{2} \\ \cos^2(\theta) &= \frac{1 + \cos(2\theta)}{2}. \end{aligned}$$

Limits. The following is proved in Math 112:

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1 \quad \text{and} \quad \lim_{x \to 0} \frac{1 - \cos(x)}{x} = 0.$$