Math 111 lecture for Friday, Week 13

Lotka-Volterra predator-prey model

The Lotka-Volterra model is a model of population growth for competing species. It was first proposed by Lotka in 1920 and used in his book on biomathematics. For Volterra's involvement, here is a quote from Wikipedia:

> The same set of equations were published in 1926 by Vito Volterra, a mathematician and physicist who had become interested in mathematical biology. Volterra's enquiry was inspired through his interactions with the marine biologist Umberto D'Ancona who was courting his daughter at the time and later was to become his son-in-law. D'Ancona studied the fish catches in the Adriatic Sea and had noticed that the percentage of predatory fish caught had increased during the years of World War I (1914–18). This puzzled him as the fishing effort had been very much reduced during the war years. Volterra developed his model independently from Lotka and used it to explain d'Ancona's observation.

Here is a sample of Anacona's data: The table shows percentages of predators found in a certain fish catch:

year	1914	1915	1916	1917	1918	1919	1920	1921	1922	1923
percentage	12	21	22	21	36	27	16	16	15	11

What might account for these data?

$$\begin{aligned} x(t) &= \text{ prey fish population} \\ y(t) &= \text{ predator fish population} \end{aligned}$$

The Lotka-Volterra equations are

$$\begin{aligned} x'(t) &= x(t)(a - by(t)) \\ y'(t) &= y(t)(-c + dx(t)). \end{aligned}$$

where a, b, c, d are positive constants. We also suppose that $x(t)$ and $y(t)$ are always positive.

Exercises.

1. What type of growth for $x'(t)$ is predicted if $y(t)$ is very small? What type of growth for $y'(t)$ is predicted if $x(t)$ is very small? How do the parameters a and c influence that growth?

2. Consider the system of equations when $a = 2$, $b = 1$, $c = 0.25$, and $d = 1$:

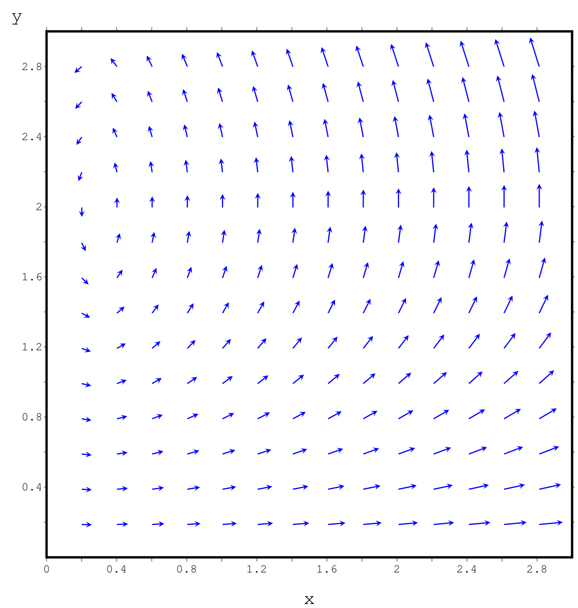
$$\begin{aligned} x'(t) &= x(t)(2 - y(t)) \\ y'(t) &= y(t)(-0.25 + x(t)). \end{aligned}$$

   Under what condition is $x'(t) = 0$? Under what condition is $y'(t) = 0$. How about both $x'(t) = 0$ and $y'(t) = 0$? What does this mean in terms of the two populations?

3. Answer the previous question for the general system (with arbitrary positive constants a, b, c, d).

Picture the Lotka-Volterra system. Here is a way of picturing the system:

$$\begin{aligned} x'(t) &= x(t)(2 - y(t)) \\ y'(t) &= y(t)(-0.25 + x(t)). \end{aligned}$$



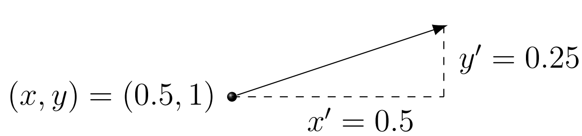
In the diagram, the horizontal axis is the prey population, x, and the vertical axis is the predator population, y. Each point (x, y) in the plane represents a potential state of the two-species system. For instance, the point $(0.5, 1)$ represents the state in which the prey population is 0.5 and the predator population is 1. (To make this more realistic, we could think of the units as being hundreds of fish.) What do the arrows mean? At each point (x, y), we have attached the vector

$$(x', y') = (x(t)(2 - y(t)),\ y(t)(-0.25 + x(t))).$$

For example, if $x = 0.5$ and $y = 1$, then

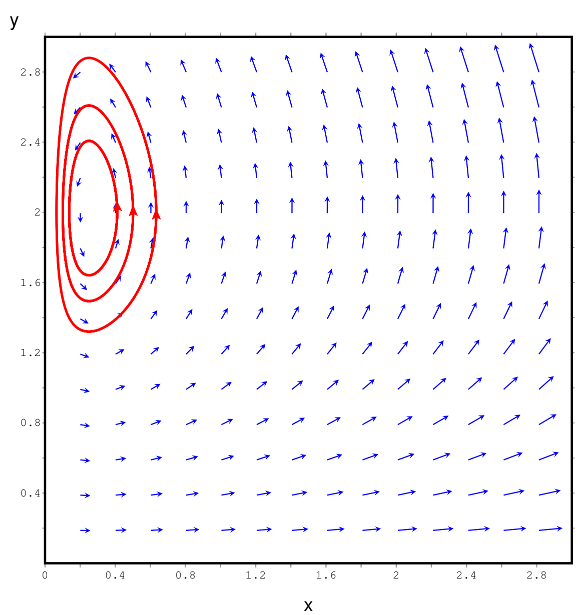
$$\begin{aligned} x' &= x(2 - y) = 0.5(2 - 1) = 0.5 \\ y' &= y(-0.25 + x) = 1(-0.25 + 0.5) = 0.25. \end{aligned}$$

So the rate of change of the prey population is 0.5 and the rate of change of the predatory population is 0.25. We picture this in the diagram as follows:



Exercise. Find the point $(0.5, 1)$ in the diagram and check that the arrow attached to that point matches the picture given above.

The below illustrates three possible solutions to the system of differential equations. The idea is to think of the system of arrows as a flowing liquid. Drop a tiny boat in the fluid and see where it flows over time. The three solutions correspond to three different initial conditions, i.e., three spots where we have dropped a tiny boat.



If you follow just the x coordinate, you will see how the prey population changes over time. If you follow the y coordinate, you will see how the pedator population changes over time.

Exercises.

1. How do the populations change over time for the three different solutions?
2. Where is the point where $x' = y' = 0$ in the picture?

General question. What are some of the unrealistic assumptions behind the model?