Math 111 lecture for Wednesday, Week 13

Differential equations

As a warm-up, let's solve the differential equation

$$y' = \frac{3t}{y}.$$

This is a *separable* differential equation, meaning that we can get the ys on one side of the equality and the ts on the other:

$$yy' = 3t.$$

We can then solve by integrating both sides with respect to t:

$$\int y(t)y'(t)\,dt = \int 3t\,dt.$$

The right-hand side is

$$\int 3t\,dt = \frac{3}{2}t^2$$

(We will add a constant "$+c$" at the end of our calculations.) For the left-hand side, make the substitution $u = y(t)$. So $du = y'(t)\,dt$. Substituting gives:

$$\int y(t)y'(t)\,dt = \int u\,du = \frac{1}{2}u^2 = \frac{1}{2}y^2.$$

Setting the two sides equal and adding a constant gives the most general solution:

$$\frac{1}{2}y^2 = \frac{3}{2}t^2 + \tilde{c}$$

or, equivalently,

$$\boxed{y(t)^2 = 3t^2 + c}$$

for some constant c.

To find a particular solution, we can impose an initial condition. For instance, if $y(0) = 5$, then

$$25 = y(0)^2 = 3 \cdot 0^2 + c \quad \Rightarrow \quad c = 25,$$

and the solution is

$$y(t)^2 = 3t^2 + 25.$$

Exponential growth and decay model. Let $y(t)$ now denote the size of a population, varying over time. What happens if we assume that the rate of growth of the population is proportional to the size of the population? The rate of growth of the population is $y'(t)$ and the size of the population if $y(t)$. To say they are proportional is to say there is a constant k such that

$$y'(t) = ky(t).$$

This is a separable equation, which is easy to solve:

$$y'(t) = ky(t) \quad \Rightarrow \quad \frac{y'(t)}{y(t)} = k \quad \Rightarrow \quad \int \frac{y'(t)}{y(t)}\,dt = \int k\,dt.$$

The right-hand side is

$$\int k\,dt = kt.$$

The left-hand side can be solved with the u-substitution $u = y(t)$ and $du = y'(y)\,dt$:

$$\int \frac{y'(t)}{y(t)}\,dt = \int \frac{du}{u} = \ln(u) = \ln(y).$$

Setting these equal and adding a constant gives:

$$\ln(y) = kt + c.$$

Exponentiate both sides of this equation:

$$y = e^{\ln x} = e^{kt+c} = e^c e^{kt}.$$

Since e^c is just some constant, we will relabel it as a to get

$$\boxed{y(t) = ae^{kt}.}$$

Setting $t = 0$, we see

$$y(0) = ae^0 = a.$$

Hence, a is the initial population.

Example. If $y(t) = ae^{kt}$, at what time t has the population doubled?

SOLUTION: The initial population size is a. So we are trying to find the time t when $y(t) = 2a$, so we need to solve

$$ae^{kt} = 2a.$$

Supposing that $a > 0$, we need to solve

$$y(t) = e^{kt} = 2$$

for t. Take logs:

$$\ln(2) = \ln(e^{kt}) = kt.$$

Hence, assuming $k \neq 0$,

$$t = \frac{\ln(2)}{k}.$$

Population model based on Newton's law of cooling. Suppose now that the rate of change of the population is governed by the differential equation

$$y'(t) = r(S - y(t))$$

where r and S are positive constants.

Questions:

1. When is the population increasing? Decreasing?

   ANSWER: We have

   $$y'(t) = r(S - y(t)) > 0 \quad \Leftrightarrow \quad S - y(t) > 0 \quad \Leftrightarrow \quad S > y(t).$$

   So the population is increasing whenever it's less than S and decreasing whenever it's larger than S.

2. What is the long-term behavior of the population?

   ANSWER: Given the answer to the previous problem it seems like the population should stabilize at S

3. Solve the equation.

   SOLUTION: The equation is separable:

   $$\begin{aligned} y'(t) = r(S - y(t)) \quad &\Rightarrow \quad \frac{y'(t)}{S - y(t)} = r \\ &\Rightarrow \quad \int \frac{y'(t)}{S - y(t)}\, dt = \int r\, dt \\ &\Rightarrow \quad \int \frac{y'(t)}{S - y(t)}\, dt = rt + c. \end{aligned}$$

Substitute $u = S - y(t)$. Then $du = -y'(t)\,dt$. So

$$\int \frac{y'(t)}{S - y(t)}\,dt = -\int \frac{du}{u} = -\ln(u) = -\ln(S - y(t)) = \ln\left((S - y(t))^{-1}\right).$$

Therefore,

$$\ln\left(\frac{1}{S - y(t)}\right) = rt + c.$$

Exponentiate:

$$\frac{1}{S - y(t)} = e^{rt+c} = e^c e^{rt} = ae^{rt} \qquad (a = e^c),$$

and solve for $y(t)$:

$$\frac{1}{S - y(t)} = ae^{rt} \quad \Rightarrow \quad S - y(t) = \frac{1}{ae^{rt}}$$

$$\Rightarrow \quad y(t) = S - \frac{1}{ae^{rt}}.$$

Therefore, the solution is

$$y(t) = S - \frac{1}{a}e^{-rt}.$$

Note that $y(t) \to S$ as $t \to \infty$.

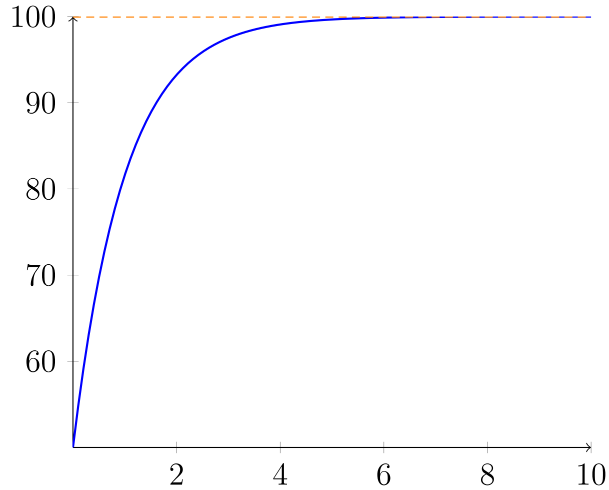
Let's now make the initial population explicit in the solution. Say I is the initial population. Then

$$I = y(0) = S - \frac{1}{a}e^0 = S - \frac{1}{a}.$$

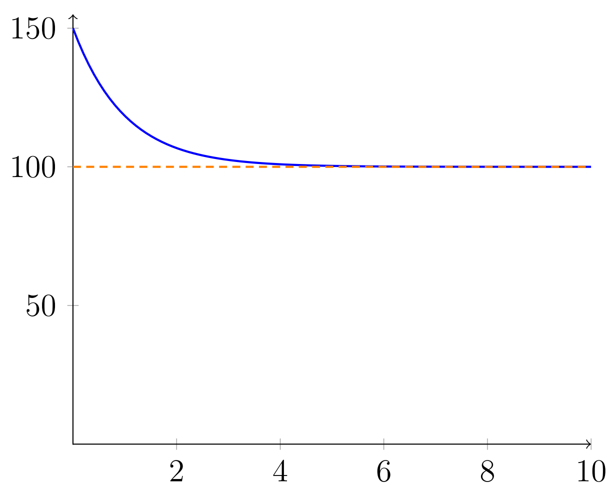
Therefore, $a = (S - I)^{-1}$. Our final form for the equation is

$$\boxed{y(t) = S + (I - S)e^{-rt},}$$

where $I = y(0)$ is the initial population.



Graph of $y(t) = S + (I - S)e^{-rt}$ with $S = 100$, $I = 50$, and $r = 1$.



Graph of $y(t) = S + (I - S)e^{-rt}$ with $S = 100$, $I = 150$, and $r = 1$.