Math 111 lecture for Monday, Week 13

Differential equations

Let y be a function of t. An equation in t and the derivates of y is called a *differential equation*. For instance, the following is a differential equation:

$$y'' - y = 0.$$

To make the dependence on t explicit, we might instead write

$$y''(t) - y(t) = 0.$$

Imagine that $y(t)$ gives the position of a particle on the real number line at time t. Then $y'(t)$ is the velocity of the particle—the rate of change of $y(t)$ over time, and $y''(t)$ is the acceleration of the particle—the rate of change of its velocity over time. The differential equation we are thinking about can be rewritten as

$$y''(t) = y(t).$$

So as the position of the particle gets larger, its acceleration increases.

To *solve* the differential equation, we need to find all functions $y(t)$ that satisfy the equation: what possible distance functions have this behavior. It turns out that the most general solution is

$$y = ae^t + be^{-t} \tag{1}$$

where a and b are any constants. To check this general y is a solution, we check that it satisfies the equation:

$$\begin{aligned}\left(ae^t + be^{-t}\right)'' - (ae^t + be^{-t}) &= \left(ae^t - be^{-t}\right)' - (ae^t + be^{-t})\\ &= \left(ae^t + be^{-t}\right) - (ae^t + be^{-t})\\ &= 0.\end{aligned}$$

We say there is a *two-parameter family of solutions*, the parameters being a and b. This turns out to be expected since the equation we are considering is a *second-order* differential equation, which means that the highest-order derivative in the equation is y'', a second derivative. To specify a particular solution, i.e., to determine the constants a and b, we need to set two initial conditions. For instance, suppose we want the solution with initial position $y(0) = 0$ and initial velocity $y'(0) = 1$. Plugging $y(0) = 0$ into equation (1), we get

$$0 = y(0) = ae^0 + be^{-0} = a + b,$$

Differentiating equation 1, we get

$$y'(t) = ae^t - be^{-t},$$

and hence, if $y'(0) = 1$, we need

$$1 = y'(0) = ae^0 - be^{-0} = a - b.$$

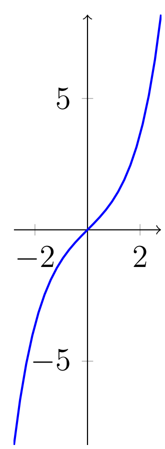
To summarize: the initial conditions force

$$\begin{aligned} a + b &= 0 \\ a - b &= 1. \end{aligned}$$

Adding the two equations gives $2a = 1$; so $a = 1/2$. Then since $a + b = 0$, we need $b = -1/2$. So the solution with initial position at the origin and initial velocity 1 is

$$y = \frac{1}{2}(e^t - e^{-t}).$$

The graph looks like this:



Graph of $y(t) = \dfrac{1}{2}(e^t - e^{-t})$.

Note that the differential equation and its initial conditions determine the movement of the particle, i.e., determine $y(t)$, for all time, not just stating from the initial time $t = 0$. The behavior is roughly like this: as t becomes large, the term e^{-t} becomes almost 0, so $y(t) \approx e^t/2$ for large t. This is what we might expect from the equation $y'' = y$. Once the position and velocity are positive, the acceleration is positive, which causes the speed to increase. The speed becomes even more positive, which causes the distance to increase, which causes the acceleration to increase, etc.

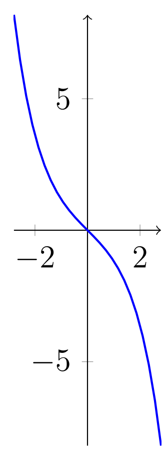
What if we had initial conditions $y(0) = 0$ and $y'(0) = -1$. This is just like before but now the initial velocity is to the left instead of to the right. Repeating the above reasoning, we find that we need to solve the system of equations

$$\begin{aligned} a + b &= 0 \\ a - b &= -1. \end{aligned}$$

Adding the equations gives $2a = -1$, this time; so $a = -1/2$. Then, since $a + b = 0$, we get $b = 1/2$. The solution is then

$$y = -\frac{1}{2}(e^t - e^{-t}).$$

The graph looks like this:



Graph of $y(t) = -\dfrac{1}{2}(e^t + e^{-t})$.

The particle speeds off in the negative direction.

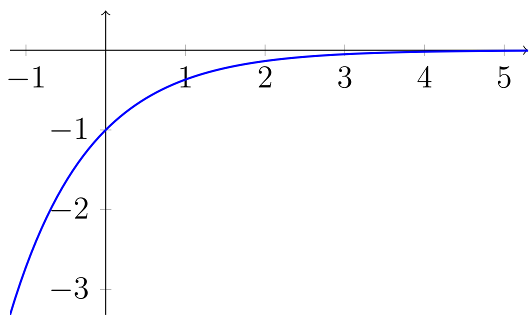
What would happen if the particle start at $y(0) = -1$, to the left of the origin, but with initial velocity $y'(0) = 1$, to the right. So the particle is moving to the origin, which according to $y'' = y$ should make the acceleration decrease. The set of equations for initial conditions $y(0) = -1$ and $y'(0) = 1$ is

$$\begin{aligned} -1 &= y(0) = ae^0 + b^{-0} = a + b \\ 1 &= y'(0) = ae^0 - b^{-0} = a - b \end{aligned}$$

or

$$
\begin{aligned}
a+b &= -1 \\
a-b &= 1.
\end{aligned}
$$

Adding the equations gives $2a = 0$, hence $a = 0$. Then $a + b = -1$ gives $b = -1$. So the solution in this case is $y(t) = -e^{-t}$.



Graph of $y(t) = -e^{-t}$.

The particle gets closer and closer to the origin, decelerating all the time and never quite getting there.

As a final example, imagine the initial condition $y(0) = y'(0) = 0$. So the particle is initially sitting at the origin with no velocity. We have $y''(t) = y$. So initially, we have $y''(0) = y(0) = 0$. So there is also no acceleration. The system of equations for the constants is

$$
\begin{aligned}
a+b &= 0 \\
a-b &= 0.
\end{aligned}
$$

Adding the equations gives $2a = 0$, hence, $a = 0$. Then the equation $a + b = 0$ implies $b = 0$, too. So the solution in this case is

$$y(t) = 0.$$

The particle just sits at the origin.

Next example. Now consider the first-order equation

$$ty'(t) = 3y(t).$$

Again, let's think of $y(t)$ of specifying the position of a particle on the real number line at time t. Suppose that $t \neq 0$ and solve for y':

$$y' = \frac{3y}{t}.$$

The t in the denominator is trying to decrease the velocity over time. On the other hand, the particle moving to the right would increase $y(t)$, which would increase the velocity. What is the resulting behavior?

This equation is known as a *separable* differential equation. That means that it is possible to get all of the ys on one side of the equation and the ts on the other:

$$\frac{y'}{y} = \frac{3}{t}.$$

Now try to integrate both sides with respect to t:

$$\int \frac{y'}{y}\,dt = \int \frac{3}{t}\,dt.$$

The right-hand side is easy:

$$3\int \frac{1}{t}\,dt = 3\ln(t) + c$$

For the left-hand side, use the substitution $u = y$. Then $du = dy$, so we get

$$\int \frac{y'}{y}\,dt = \int \frac{du}{u}\,du = \ln(u) + \tilde{c} = \ln(y) + \tilde{c}.$$

Setting the two sides equal gives

$$\ln(y) = 3\ln(t) + k = \ln(t^3) + k$$

for some constant k. Exponentiate to get

$$e^{\ln(y)} = e^{\ln(t^3)+k} = e^k e^{\ln(t^3)},$$

and hence,

$$y = Kt^3.$$

The solution with initial condition $y(1) = 1$ is $y(t) = t^3$.