Math 111 lecture for Friday, Week 13

Logistic growth model. Let $P(t)$ be the size of a population at time t. The logistic growth model is the differential equation

$$P'(t) = rP(t)\left(1 - \frac{P(t)}{K}\right).$$

It says the growth in population is proportional to the size of the existing population with an extra factor to account for limited resources. When the population is small (when P is much smaller then K), we see $P' \approx rP$, which we've already seen leads to exponential growth. However, as P gets close to K over time, the factor $1 - P/K$ slows the growth.

Solution. The equation is separable and can be solved using integration using the technique of partial fractions.

$$P'(t) = rP(t)\left(1 - \frac{P(t)}{K}\right) \quad \Rightarrow \quad \frac{P'(t)}{P(t)\left(1 - \frac{P(t)}{K}\right)} = r.$$

The technique of partial fractions requires us to find constants A and B such that

$$\frac{1}{P(t)\left(1 - \frac{P(t)}{K}\right)} = \frac{A}{P(t)} + \frac{B}{1 - \frac{P(t)}{K}}. \tag{1}$$

We have

$$\frac{A}{P(t)} + \frac{B(t)}{1 - \frac{P(t)}{K}} = \frac{A\left(1 - \frac{P(t)}{K}\right) + BP(t)}{P(t)\left(1 - \frac{P(t)}{K}\right)}. \tag{2}$$

Comparing numerators in equations (1) and (2), we need to adjust A and B so that

$$1 = A\left(1 - \frac{P(t)}{K}\right) + BP(t).$$

Or, rearranging:

$$1 = A + \left(-\frac{A}{K} + B\right)P(t).$$

We get an equality if

$$A = 1 \quad \text{and} \quad -\frac{A}{K} + B = 0.$$

So $A = 1$ and $B = 1/K$. Therefore we can write (double-check!):

$$\frac{1}{P(t)\left(1 - \frac{P(t)}{K}\right)} = \frac{1}{P(t)} + \frac{1/K}{1 - \frac{P(t)}{K}}. \tag{3}$$

Back to solving the differential equation:

$$\frac{P'(t)}{P(t)\left(1 - \frac{P(t)}{K}\right)} = r \quad \Rightarrow \quad \int \frac{P'(t)}{P(t)\left(1 - \frac{P(t)}{K}\right)}\, dt = \int r\, dt$$

$$\Rightarrow \quad \int \frac{P'(t)}{P(t)\left(1 - \frac{P(t)}{K}\right)}\, dt = rt + \text{constant}.$$

For the left-hand side, use equation (3):

$$\begin{aligned}
\int \frac{P'(t)}{P(t)\left(1 - \frac{P(t)}{K}\right)}\, dt &= \int \frac{P'(t)}{P(t)} + \frac{P'(t)/K}{1 - \frac{P(t)}{K}}\, dt \\
&= \int \frac{P'(t)}{P(t)}\, dt + \frac{1}{K} \int \frac{P'(t)}{1 - \frac{P(t)}{K}}\, dt \\
&= \ln P(t) + \frac{1}{K} \int \frac{P'(t)}{1 - \frac{P(t)}{K}}\, dt
\end{aligned}$$

For the remaining integral, let $u = 1 - P(t)/K$. Then $du = -\frac{1}{K} P'(t)\, dt$, and $-K\, du = P'(t)\, dt$. Therefore,

$$\begin{aligned}
\frac{1}{K} \int \frac{P'(t)}{1 - \frac{P(t)}{K}}\, dt &= \frac{1}{K} \int \frac{-K}{u}\, du \\
&= -\int \frac{du}{u} \\
&= -\ln(u) + \text{constant} \\
&= -\ln\left(1 - \frac{P(t)}{K}\right) + \text{constant}.
\end{aligned}$$

Putting this all together:

$$\begin{aligned}
\ln P(t) - \ln\left(1 - \frac{P(t)}{K}\right) &= \ln P(t) + \ln\left(\left(1 - \frac{P(t)}{K}\right)^{-1}\right) \\
&= \ln\left(P(t)\left(1 - \frac{P(t)}{K}\right)^{-1}\right) \\
&= rt + \text{constant}.
\end{aligned}$$

Exponentiate both sides to get

$$P(t)\left(1 - \frac{P(t)}{K}\right)^{-1} = e^{rt}e^{\text{constant}} = ae^{rt}$$

for some positive constant a. We now need to solve this equation for $P(t)$:

$$\begin{aligned}
ae^{rt} &= P(t)\left(1 - \frac{P(t)}{K}\right)^{-1} = \frac{KP(t)}{K - P(t)} \\
&\Rightarrow ae^{rt}(K - P(t)) = KP(t) \\
&\Rightarrow aKe^{rt} = ae^{rt}P(t) + KP(t) = (ae^{rt} + K)P(t) \\
&\Rightarrow P(t) = \frac{aKe^{rt}}{ae^{rt} + K} \\
&\Rightarrow P(t) = \frac{aK}{a + Ke^{-rt}}.
\end{aligned}$$

We would like to express the arbitrary constant a in terms of the initial population:

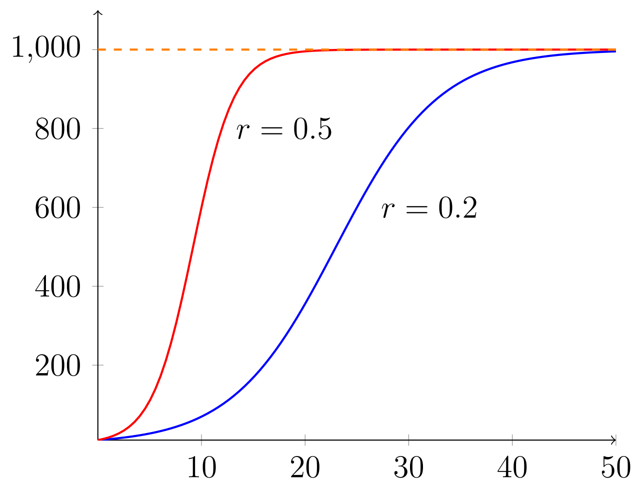
$$\begin{aligned}
&P(0) = \frac{aKe^0}{ae^0 + K} = \frac{aK}{a + K} \\
&\Rightarrow P(0)(a + K) = aK \\
&\Rightarrow P(0)K = aK - P(0)a = a(K - P(0)) \\
&a = \frac{P(0)K}{K - P(0)}.
\end{aligned}$$

Substituting this expression for a and simplifying gives the final form for the solution

$$\boxed{P(t) = \frac{P(0)K}{P(0) + (K - P(0))e^{-rt}}.}$$

It's easy to see from this equation that the limiting population is

$$\lim_{t \to \infty} P(t) = K.$$



Graph of $P(t)$ with $K = 1000$ and $P(0) = 10$ and two different growth rates: $r = 0.5$ in red and $r = 0.2$ in blue.

Exercise. A state game commission releases 40 elk into a game refuge. After 5 years, the elk population is 104. The commission believes that the refuge can support no more than 4000 elk. Use a logistic model to predict the elk population in 15 years.

SOLUTION: The carrying capacity is $K = 4000$, so the logistic model in this situation is

$$P'(t) = rP(t)\left(1 - \frac{P(t)}{4000}\right)$$

where we can determine r from the additional information we're given. The initial population size is $P(0) = 40$. From the solution to the logistic equation we derived above, we have

$$P(t) = \frac{4000P(0)}{P(0) + (4000 - P(0))e^{-rt}}$$

$$= \frac{160000}{40 + 3960e^{-rt}}$$

$$= \frac{4000}{1 + 99e^{-rt}}$$

We are given that $P(5) = 104$. Therefore,

$$104 = P(5) = \frac{4000}{1 + 99e^{-5r}}.$$

Solve for r:

$$\begin{aligned}
104 = \frac{4000}{1 + 99e^{-5r}} \quad &\Rightarrow \quad 104(1 + 99e^{-5r}) = 4000 \\
&\Rightarrow \quad e^{-5r} = \frac{1}{99}\left(\frac{4000}{104} - 1\right) = \frac{487}{1287} \\
&\Rightarrow \quad -5r = \ln\left(\frac{487}{1287}\right) \\
&\Rightarrow \quad r \approx 0.194.
\end{aligned}$$

So our model for this population is

$$P(t) = \frac{4000}{1 + 99e^{-0.194\,t}}$$

So we would predict the population after 15 years to be

$$P(15) = \frac{4000}{1 + 99e^{-0.194 \cdot 15}} \approx 626.$$