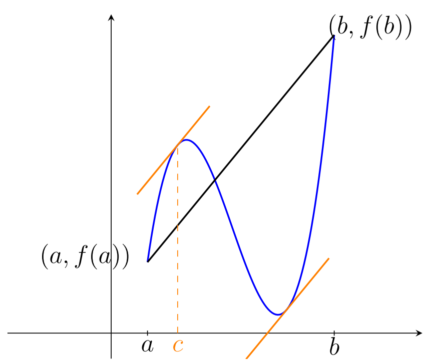
Math 111 lecture for Monday, Week 12

Mean Value Theorem

Suppose you drive through two toll booths that are 80 miles apart. Your time is recorded at each booth, and it is determined that it took you 1 hour to travel that distance. Why would it be reasonable for you to be issued a ticket? The mathematical basis for your answer is the following theorem:

Mean Value Theorem (MVT). Let f be a continuous function on $[a, b]$ and differentiable on (a, b). Then there exists a number c with $a < c < b$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$



Mean value theorem (with two possible choices for c).

Proof. Math 112. □

If we think of f as a distance function, the MVT says that at some point between a and b, the instantaneous speed is equal to the average speed over the time interval $[a, b]$.

The MVT is a *local-to-global* tool: the instantaneous speed only depends on the behavior of f near c, while the average speed depends on the net behavior of f over the entire interval. In more detail, recall the definition of the derivative:

$$f'(c) = \lim_{x \to \infty} \frac{f(c + h) - f(c)}{h}.$$

If you think of the definition of the limit, it's clear that $f'(c)$ only depends on the behavior of f in a tiny interval around c.

Fundamental applications of the MVT. Consider the following statement:

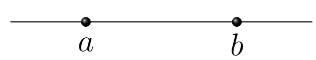
If $f'(x) = 0$ for every real number x, then f is a constant function.

How would you go about proving this from the definition of the derivative? A similar statement: if $f'(x) > 0$ for all x in an interval, then f is increasing on that interval. We will prove these now as corollaries of the MVT.

Corollary. Let f be a differentiable function on an open interval I. Then:

1. If $f'(x) = 0$ for all x in I, then f is constant on I.
2. If $f'(x) > 0$ for all x in I, then f is increasing on I.
3. If $f'(x) < 0$ for all x in I, then f is decreasing on I.

Proof. Let a and b be in the interval I, and suppose $a < b$:

$I =$ 

By the MVT, there exists c between a and b such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}. \tag{1}$$

1. If $f' = 0$ at all points in I. Then, in particular, $f'(c) = 0$. Therefore, equation (1) becomes
$$0 = \frac{f(b) - f(a)}{b - a},$$
which implies $f(a) = f(b)$. Since a, b are arbitrary points in I, it follows that f is constant on I.

2. If $f' > 0$ at all points in I, then equation (1) becomes
$$0 < f'(c) = \frac{f(b) - f(a)}{b - a},$$
which implies $f(a) < f(b)$. Since a, b are arbitrary points in I, it follows that f is increasing on I.

3. If $f' < 0$ at all points in I, a similar argument shows that f is decreasing on I. $\square$