Math 111 lecture for Monday, Week 11

Another integration by parts example. Recall the formula for integration by parts:

$$\int u\,dv = uv - \int v\,du.$$

Let's apply this to compute the indefinite integral

$$\int e^x \cos(x)\,dx.$$

Let

$$\begin{array}{ll} u = e^x & du = e^x\,dx \\ dv = \cos(x)\,dx & v = \sin(x). \end{array}$$

We get

$$\int e^x \cos(x)\,dx = e^x \sin(x) - \int e^x \sin(x)\,dx. \tag{1}$$

So we now need to compute $\int e^x \sin(x)\,dx$. Try again by parts, this time with

$$\begin{array}{ll} u = e^x & du = e^x\,dx \\ dv = \sin(x)\,dx & v = -\cos(x). \end{array}$$

We get

$$\int e^x \sin(x)\,dx = -e^x \cos(x) + \int e^x \cos(x)\,dx.$$

It may look like we've gone in circles. But if you carefully substitute what we've just calculated back into equation (1), we get

$$\begin{aligned} \int e^x \cos(x)\,dx &= e^x \sin(x) - \int e^x \sin(x)\,dx \\ &= e^x \sin(x) - (-e^x \cos(x) + \int e^x \cos(x)\,dx) \\ &= e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x)\,dx). \end{aligned}$$

Add $\int e^x \cos(x)\,dx$ to both sides to get

$$2\int e^x \cos(x)\,dx = e^x \sin(x) + e^x \cos(x).$$

Therefore, the final solution is

$$\int e^x \cos(x)\,dx = \frac{1}{2}(\sin(x)+\cos(x))e^x + c.$$

(It's easy to take the derivative of the right-hand side, using the product rule, to check that we've got the right answer.)

To define the logarithm function, we first need to talk about a theorem sometimes called the second fundamental theorem of calculus. It says that every continuous function has an antiderivative.

Theorem. (FTC2) Suppose f is a continuous function on an open interval I containing a point a. Define

$$g(x) = \int_a^x f(t)\,dt$$

for each $x \in I$. Then $g'(x) = f(x)$ for each $x \in I$.

Proof. Math 112. $\square$

Example. Let $f(x) = x^5$ and define

$$g(x) = \int_0^x f(t)\,dt = \int_0^x t^5\,dt.$$

Therefore,

$$\begin{aligned} g(x) &= \int_0^x t^5\,dt \\ &= \frac{1}{6}t^6\bigg|_{t=0}^{x} \\ &= \frac{1}{6}x^6, \end{aligned}$$

and we have $g'(x) = f(x)$, as claimed. (Exercise: check that if we let $g(x) = \int_a^x x^5\,dt$, for any constant a, we'd still get $g'(x) = f(x)$.)

Example. Here is an example of an integral that does not have a nice closed form:

$$b(x) = \int_0^x e^{-t^2+\cos(t)}\,dt.$$

However, by FTC2, we know its derivative

$$b'(x) = e^{-x^2+\cos(x)}.$$

The logarithm function. Recall that for every real number α, we have $(x^\alpha)' = \alpha x^{\alpha-1}$. Therefore,

$$\int x^\alpha \, dx = \frac{1}{\alpha+1} x^{\alpha+1} + c$$

as long as $\alpha \neq 1$. So the case $\alpha = -1$ is somewhat mysterious:

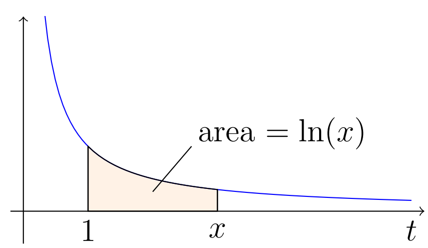
$$\int x^{-1} \, dx = \int \frac{1}{x} \, dx.$$

We handle this case by making up a name for it—the logarithm:

Definition. For $x > 0$, the *natural logarithm* is

$$\ln(x) = \int_1^x \frac{1}{t} \, dt.$$

So the natural logarithm is, by definition, the area under the graph of $f(t) = 1/t$ from $t = 1$ to $t = x$:



Graph of $f(t) = \frac{1}{t}$.

By FTC2, we have

$$(\ln(x))' = \frac{1}{x}.$$