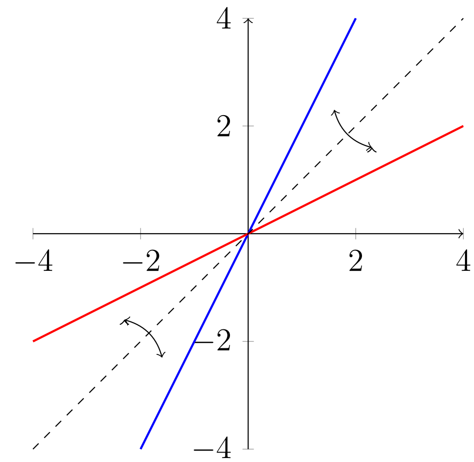
Math 111 lecture for Friday, Week 11

The inverse function theorem and exponentials

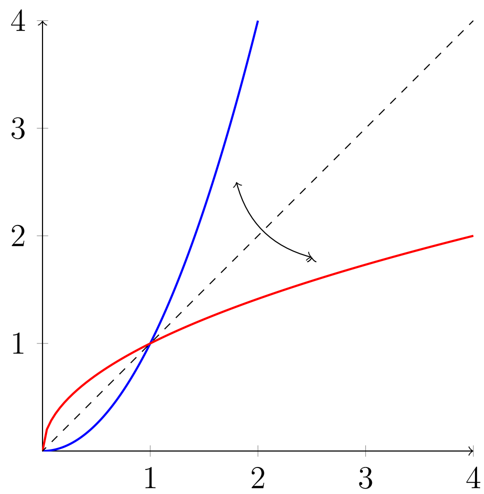
Definition. Functions f and g are inverses of each other if

$$f(g(x)) = x \quad \text{and} \quad g(f(x)) = x.$$

Examples.



Graphs of inverse functions $f(x) = 2x$ and $g(x) = \frac{1}{2}x$.

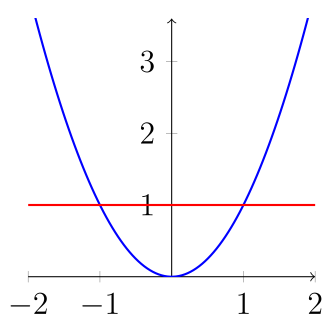


Graphs of inverse functions $f(x) = x^2$ and $g(x) = \sqrt{x}$.

Definition. A function f is *one-to-one* if $x \neq y$ implies $f(x) \neq f(y)$.

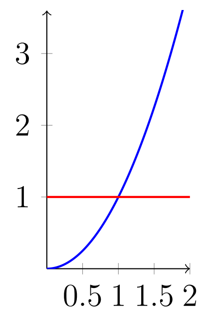
So f is one-to-one if it does not send two points to the same point. Graphically, this means that no horizontal line will meet the graph of f more than once.

Example. The function $f(x) = x^2$ as a function of the whole real number line is not one-to-one. For instance $f(1) = f(-1)$. Graphically, there exist horizontal lines meeting the graph of f in more than one point:



$f(x) = x^2$ fails the horizontal line test on $(-\infty, \infty)$.

However, if we restrict $f(x) = x^2$ to be a function on $[0, \infty)$, it is one-to-one:



$f(x) = x^2$ passes the horizontal line test on $[0, \infty)$.

Proposition. If the function f is one-to-one, it has an inverse.

Example. Considering $f(x) = x^2$ as a function on $[0, \infty)$, then it has an inverse: $g(x) = \sqrt{x}$.

Theorem. (Inverse function theorem, (IFT).) Suppose f is differentiable, and suppose f has an inverse g. Then g is differentiable and

$$g'(x) = \frac{1}{f'(g(x)}$$

provided $f'(g(x)) \neq 0$.

Proof. We can give a proof of part of this theorem. Suppose g is differentiable. Since f and g are inverses, we have $f(g(x)) = x$. Take derivatives and apply the chain rule:

$$1 = (x)' = (f(g(x))' = f'(g(x))g'(x).$$

So $1 = f'(g(x))g'(x)$. Solve for $g'(x)$ to get

$$g'(x) = \frac{1}{f'(g(x))}.$$

$\square$

Example. Let's check the IFT with an example. The functions $f(x) = x^2$ and $g(x) = \sqrt{x}$ are inverse functions on $[0, \infty)$. We have

$$g'(x) = (x^{1/2})' = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}.$$

Let's compare this with $1/f'(g(x))$. We have $f'(x) = 2x$, and $g(x) = \sqrt{x}$. So

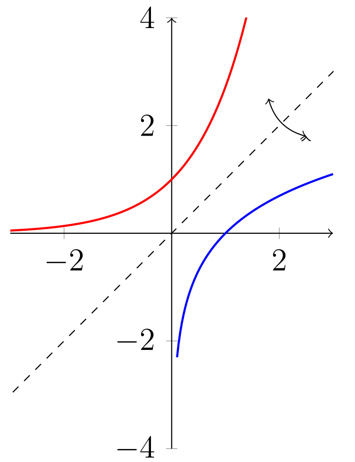
$$\frac{1}{f'(g(x))} = \frac{1}{f'(\sqrt{x})} = \frac{1}{2\sqrt{x}}.$$

The exponential function. Recall that $\ln(x)' = 1/x > 0$ for all $x > 0$. Thus, $\ln(x)$ is always increasing. In particular, this means that $\ln(x)$ has an inverse function. By definition, the *exponential function*,

$$\exp(x)$$

is the inverse of $\ln(x)$. In other words,

$$\exp(\ln(x)) = \ln(\exp(x)) = x.$$



Graphs of inverse functions $\ln(x)$ and $\exp(x)$.

Properties of the exponential function.

1. $\exp(0) = 1$.

   Proof. Since $\ln(1) = 0$ and $\exp(\ln(x)) = x$ for all $x > 0$, we have

   $$\exp(0) = \exp(\ln(1)) = 1.$$

   □

2. $\exp(x+y) = \exp(x)\exp(y)$.

   Proof. Recall that $\ln(xy) = \ln(x) + \ln(y)$ for all $x, y > 0$. Therefore,

   $$\ln(\exp(x)\exp(y)) = \ln(\exp(x)) + \ln(\exp(y)) = x + y.$$

   So $\ln(\exp(x)\exp(y)) = x+y$. Apply the exp function to both sides and use the fact that it is inverse to ln:

   $$\begin{aligned} \ln(\exp(x)\exp(y)) = x + y \quad &\Longrightarrow \quad \exp(\ln(\exp(x)\exp(y))) = \exp(x+y) \\ &\Longrightarrow \quad \exp(x)\exp(y) = \exp(x+y). \end{aligned}$$

   □

3. $\exp'(x) = \exp(x)$.

   Proof. This follows from the inverse function theorem, but we can see it directly from the chain rule:

   $$\begin{aligned} \ln(\exp(x)) = x \quad &\Longrightarrow \quad (\ln(\exp(x)))' = (x)' \\ &\Longrightarrow \quad \frac{1}{\exp(x)}\exp'(x) = 1 \\ &\Longrightarrow \quad \exp'(x) = \exp(x). \end{aligned}$$

   □

The number e and exponentiation. We define the number e, Euler's constant, as follows:

$$e := \exp(1).$$

We would like to show that from this one simple definition it follows that

$$e^x = \exp(x) \tag{1}$$

for *all* real numbers x. The problem arises is understanding what is meant by taking a number to a power. We break this up into cases:

Case 1. For the exponent $n = 0$, we take $e^0 = 1$, by definition. We have already seen that since $\ln(1) = 0$, we have $\exp(0) = 1$. So in that case, equation (1) holds: $e^0 = \exp(0)$.

Case 2. For the exponent $n = 1$, we have $e^1 = e$, by definition of exponentiation, and we have $e = \exp(1)$, by definition of e. So the equation holds here, too.

Case 3. Suppose $n = 2, 3, \ldots$ Here, we repeatedly use the fact we saw earlier: since the logarithm converts products to sums, $\ln(xy) = \ln(x) + \ln(y)$, it follows that the exponential function converts sums to products, $\exp(x+y) = \exp(x)\exp(y)$. It follows that

$$
\begin{aligned}
&e^2 = e \cdot e = \exp(1)\exp(1) = \exp(1+1) = \exp(2) \\
&e^3 = e \cdot e \cdot e = \exp(1)\exp(1)\exp(1) = \exp(1+1+1) = \exp(3) \\
&e^4 = e \cdot e \cdot e \cdot e = \exp(1)\exp(1)\exp(1)\exp(1) = \exp(1+1+1+1) = \exp(4),
\end{aligned}
$$

and so on.

Case 4. What about negative exponents? For $n = 1, 2, 3, \ldots,$ we would like to show that $e^{-n} = \exp(-n)$. First: what does e^{-n} mean? Answer: by definition

$$e^{-n} = 1/e^n.$$

Substituting in the definition of e, then, what we need to show is that

$$\exp(-n) = \frac{1}{e^n} = \frac{1}{\exp(n)}.$$

(The second equality above follows since we have already established that $e^n = \exp(n)$ for $n = 0, 1, 2, \ldots$) Here is a nice argument to establish that fact (recalling that $\exp(0) = 1$):

$$1 = \exp(0) = \exp(n - n) = \exp(n + (-n)) = \exp(n)\exp(-n).$$

So $1 = \exp(n)\exp(-n)$, and the result follows.

Case 5. What about rational exponents? Consider the fraction a/b where a and b are integers. By definition, $e^{a/b}$ is the number such that

$$(e^{a/b})^b = e^a.$$

For instance, multiplying $e^{1/2}$ by itself gives e. For a warm-up, we will show that $e^{1/2} = \exp(1/2)$. This just means that we need to show that multiplying $\exp(1/2)$ by itself should give e. We get that from the following calculation (again involving the formula $\exp(x+y) = \exp(x)\exp(y)$):

$$\exp(1/2)\exp(1/2) = \exp(1/2 + 1/2) = \exp(1) = e.$$

What about $e^{2/5}$, the number which when multiplied by itself 5 times gives e^2. Here we have

$$\begin{aligned}\exp(2/5)\exp(2/5)\exp(2/5)\exp(2/5)\exp(2/5) &= \exp(2/5+2/5+2/5+2/5+2/5)\\ &= \exp(2)\\ &= e^2.\end{aligned}$$

The last step follows from Case 3, above. We have just shown that $\exp(2/5) = e^{2/5}$. In general, for an arbitrary fraction a/b, we have

$$\begin{aligned}\underbrace{\exp(a/b)\exp(a/b)\cdots\exp(a/b)}_{b\text{ times}} &= \exp(\underbrace{a/b+\cdots+a/b}_{b\text{ times}})\\ &= \exp(a)\\ &= e^a.\end{aligned}$$

This shows that $\exp(a/b) = e^{a/b}$: multiplying the number $\exp(a/b)$ by itself b times gives e^a.

Case 6. The final case is where x is an arbitrary real number. We have seen that $e^x = \exp(x)$ when x is an integer or a fraction. Is it true that $e^x = \exp(x)$ when x is an irrational number (those are the only real numbers we haven't considered). The problem here is: what is the definition of e^x when x is an irrational number? What does $e^{\sqrt{2}}$ or e^π mean? There is a good chance that you have not seen a definition of exponentiation by an irrational number. We make the following definition:

Definition. Let x be any real number, and let $e = \exp(1)$. Then we define e^x by

$$e^x := \exp(x).$$

The utility of this definition is that for cases 1–5, where we have a prior notion of the meaning of e^x, this definition agrees with the usual definition, but then it extends the meaning of exponentiation to the case of irrationals, as well.

To finish the story, we define exponentiation, in general:

Definition. Let a and x be real numbers with $a > 0$. Then

$$a^x := e^{x\ln(a)} = \exp(x\ln(a)).$$

Example. $2^\pi = e^{\pi \ln(2)} \approx 8.82$.

Some consequences of the definition:

For all real numbers x and y,

- $a^{x+y} = a^x a^y$,
- $(a^x)^y = a^{xy}$,
- $\dfrac{a^x}{a^y} = a^{x-y}$,
- $\ln(a^x) = x \ln(a)$.