Math lecture for Wednesday, Week 10

Properties of integrals.

• Suppose that f and g are integrable on $[a, b]$ and $c \in \mathbb{R}$. then

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g \quad \text{and} \quad \int_a^b cf = c\int_a^b f.$$

Example.

$$\int_0^2 2x^2 - 7\cos(x)\,dx = 2\int_0^2 x^2\,dx - 7\int_0^2 \cos(x)\,dx.$$

• Suppose that f and g are integrable on $[a, b]$ and $f(x) \leq g(x)$ for all $x \in [a, b]$. Then $\int_a^b f \leq \int_a^b g$. In other words, integration preserves inequalities.

Example. We have $5\cos(x) \leq 5$ on $[3, 8]$. Therefore,

$$\int_3^8 \cos(x)\,dx \leq \int_3^8 5\,dx.$$

• Suppose f is integrable on $[a, b]$ and $a < c < b$. Then

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Example.

$$\int_0^{10} e^{3x}\,dx = \int_0^4 e^{3x}\,dx + \int_4^{10} e^{3x}\,dx.$$

• If f is continuous on $[a, b]$, then $\int_a^b f$ exists. In general,

$$\text{differentiable} \quad \Longrightarrow \quad \text{continuous} \quad \Longrightarrow \quad \text{integrable}.$$

Example. The function $f(x) = |x|$ is continuous but is not differentiable at $x = 0$. The function

$$g(x) = \begin{cases} x & \text{if } x \neq 0, \\ 5 & \text{if } x = 0. \end{cases}$$

us integrable but not continuous.

The proofs for all of the above are left to Math 112.

Practice with the FTC. Recall the FTC: Roughly, if f is integrable and $g' = f$, then

$$\int_a^b f = g(x)\Big|_a^b := g(b) - g(a).$$

For example,

$$\begin{aligned}\int_1^2 (3x^2 + 4x - 5) &= (x^3 + 2x^2 - 5x)\Big|_1^2 \\ &= (2^3 + 2 \cdot 2^2 - 5 \cdot 2) - (1^3 + 2 \cdot 1^2 - 5 \cdot 1) \\ &= 6 - (-2) = 8.\end{aligned}$$

The properties of the integral introduced above allow you to break up the integral into simpler pieces:

$$\int_1^2 (3x^2 + 4x - 5) = 3\int_1^2 x^2 + 4\int_1^2 x - 5\int_1^2 1 = \text{ etc.}$$

Compute the following problems using the FTC:

1. $\int_0^3 (2e^x + 5x)\,dx$.
2. $\int_0^\pi \sin(x)\,dx$.
3. $\int_0^2 (3x^2 + 2x + 1)(x^3 + x^2 + x)^{99}\,dx$.

Antiderivatives. The key to using the FTC to compute an integral $\int_a^b f$ is to find a function g such that $g' = f$. Such a function g is called an *antiderivative* of f.

Example. The function $x^3/3$ is an antiderivative of $f(x) = x^2$. The most general antiderivative of f is $x^3/3 + c$ where is c is any constant. We use the following notation:

$$\int x^2 = \frac{x^3}{3} + c$$

or

$$\int x^2\,dx = \frac{x^3}{3} + c.$$

Compare the above with

$$\int_0^1 x^2\,dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3}.$$

The integral $\int_0^1 x^2\,dx$ is what we have defined using lower and upper sums. It is sometimes called the *definite* integral to distinguish it from the indefinite integral $\int x^2\,dx$, which is just notation for a general antiderivative. That antiderivative can then be used to evaluate the definite integral via the FTC.

Compute the following indefinite integrals:

1. $\int x^n\,dx$ for $n = 1, 2, 3, \ldots$
2. $\int x^{1/2}\,dx$.
3. $\int x^{3/2}\,dx$.
4. $\int 5x^7 + 2x^3 + 4\,dx$.
5. $\int \cos(x) + 3e^x\,dx$.
6. $\int e^{ax}$ where a is a nonzero constant.
7. $\int \cos(4x)\,dx$.
8. $\int x(3x^2+5)^{100}\,dx$.
9. $\int x^2 e^{x^3}\,dx$.