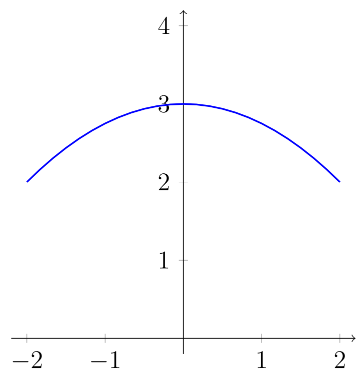
Math 111 lecture for Monday, Week 10

Fundamenta theorem of calculus

In homework, we considered the function

$$f(x) = -\frac{1}{4}x^2 + 3$$

on the interval $[-2, 2]$:



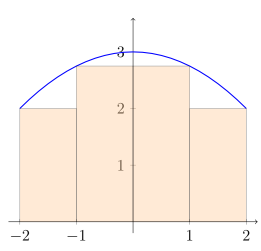
Graph of $f(x) = -\frac{1}{4}x^2 + 3$.

To estimate the integral, $\int_{-2}^{2} f$, we took the partition

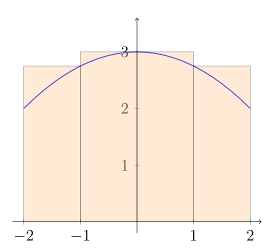
$$Q = \{-2, 1, 0, 1, 2\}$$

and found

$$L(f, Q) = 9.5 < 11.5 = U(f, Q).$$



$L(f, Q)$



$U(f, Q)$

The actual value of the integral is $\int_{-2}^{2} f = 32/3 \approx 10.7$. Averaging the upper and lower sums gives an OK approximation (given how few points are in the partition).

It is possible compute the *precise* value of the integral, $\int_{-2}^{2} f$, straight from the definition of the integral, but rather painful. To outline what needs to be done, imagine dividing the interval $[-2, 2]$ into n parts, each part with length $4/n$. That gives a partition P_n for each $n = 1, 2, 3, \ldots$. (We did something similar earlier for the function $g(x) = 2x$). Then,

$$L(f, P_n) < L\int_{-2}^{2} f \leq U\int_{-2}^{2} f < U(f, P_n).$$

Using some algebra, we can find formulas for these lower and upper sums, depending on n. From the formulas, we would then compute limits and find

$$\lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} U(f, P_n) = \frac{32}{3}.$$

which implies the upper and lower *integrals* are both equal to this limit, and thus, we have computed the integral.

There is a much easier way to compute the integral, though. It comes from the following magic known as the fundamental theorem of calculus:

Fundamental theorem of calculus (FTC). Let f be an integrable function on the interval $[a, b]$. Suppose there is a function g which is continuous on $[a, b]$ and differentiable on (a, b) such that $g' = f$. Then

$$\int_{a}^{b} f = g(b) - g(a).$$

In other words,

$$\int_{a}^{b} g' = g(b) - g(a).$$

To make the name of the variable explicit, you could write the FTC as follows:

$$\int_{a}^{b} f(x)\, dx = g(b) - g(a).$$

Definition. An *antiderivative* of a function f is any function g such that $g' = f$.

So the FTC allows us to compute the integral of a function on an interval $[a, b]$ by simply computing the *net change in the function's antiderivative over the interval.*

Example. Consider the earlier example

$$f(x) = -\frac{1}{4}x^2 + 3$$

on the interval $[-2, 2]$. To integrate f using the FTC, we first find a function g whose derivative is f. We can take

$$g(x) = -\frac{1}{12}x^3 + 3x.$$

(Every other possibility differs from the g we have chosen by the addition of a constant, and that does not affect the answer.) The FTC then says

$$\begin{aligned}
\int_{-2}^{2} f &= g(2) - g(-2) \\
&= \left(-\frac{1}{12}(2)^3 + 3(2)\right) - \left(-\frac{1}{12}(-2)^3 + 3(-2)\right) \\
&= \left(-\frac{2}{3} + 6\right) - \left(\frac{2}{3} - 6\right) \\
&= -\frac{4}{3} + 12 \\
&= \frac{32}{3}.
\end{aligned}$$

Notation. We rewrite the previous calculation using some convenient notation:

$$\int_{-2}^{2} \left(-\frac{1}{4}x^3 + 4\right) dx = \left(-\frac{1}{12}(x)^3 + 3(x)\right)\Bigg|_{-2}^{2}$$
$$= \left(-\frac{1}{12}(2)^3 + 3(2)\right) - \left(-\frac{1}{12}(-2)^3 + 3(-2)\right)$$
$$= \frac{32}{3}.$$

The big vertical line at the end of the first line of the calculation indicates that we should evaluate the function preceding the vertical line at the two endpoints, 2 and -2, then subtract.

Example. Now consider the function $f(x) = 2$. The area under $f(x)$ and above the interval $[1, 4]$ is the area of a rectangle of height 2 and base $4-1 = 3$. Hence, the area is $2 \cdot 3 = 6$. Using the FTC to solve the same problem, we first find an anti-derivative of $f(x) = 2$. For example, we could take the function $2x$. Applying the FTC, we find

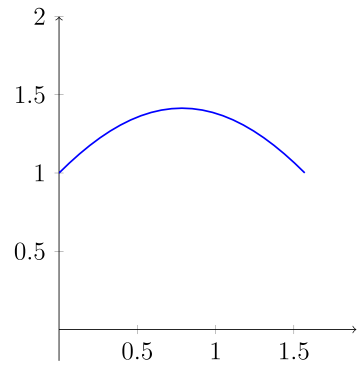
$$\int_{1}^{4} 2\, dx = 2x\Big|_{1}^{4} = 2 \cdot 4 - 2 \cdot 1 = 6.$$

Practice.[1] Compute the following integrals using the FTC:

1. $f(x) = x^3$ on $[0, 2]$.
2. $f(x) = x^3$ on $[1, 2]$.
3. $f(x) = \cos(x)$ on $[0, \pi/2]$.
4. $f(x) = \cos(x) + 1$ on $[0, \pi/2]$.
5. $f(x) = \cos(x) + \sin(x)$ on $[0, \pi/2]$.

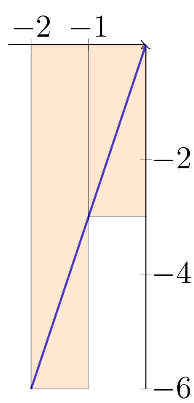
Here is a graph of the last function in the above list:

[1] Solutions appear at the end of the lecture.

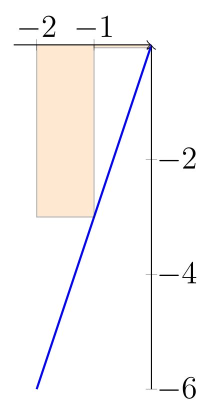


Graph of $f(x) = \cos(x) + \sin(x)$.

Area below the x-axis is counted negatively. We have been referring to the value computed by the integral as the *area* under the graph of a function. It turns out, though, that if the graph is below the x-axis, then the area *above* the graph and below the x-axis is counted negatively. For example, consider the function $f(x) = 3x$ on the interval $[-2, 0]$. Compute the upper and lower sums for f with respect to the partition $[-2, 1, 0]$:



$L(f, P) = -6 - 3 = -9$

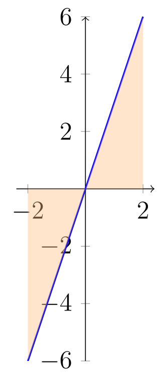


$U(f, P) = -3 - 0 = -3.$

The actual area between the graph and the x-axis is 6, but the integral will count this as -6:

$$\int_{-2}^{0} 3x\,dx = \frac{3}{2}x^2\bigg|_{-2}^{0} = \frac{3}{2}\cdot 0^2 - \frac{3}{2}\cdot -2^2 = -6.$$

What about integrating $f(x) = 3x$ on $[-2, 2]$?



Graph of $f(x) = 3x$.

The area above the axis is counted positively and the area below the axis is counted negatively, so they cancel in the integral:

$$\int_{-2}^{2} 3x\,dx = \frac{3}{2}x^2\bigg|_{-2}^{2} = \frac{3}{2}\cdot 2^2 - \frac{3}{2}\cdot -2^2 = 0.$$

If you want the actual area between the graph of f and the x-axis, you would need to break the integral into two parts—one for the area above the axis, and one for the area below the graph—and subtract:

$$\int_{0}^{2} 3x\,dx - \int_{-2}^{0} 3x\,dx = 6 - (-6) = 12.$$

Solutions to practice exercises.

1.
$$\int_0^2 x^3 = \frac{1}{4}x^4\bigg|_0^2 = \frac{1}{4}(2^4 - 0^4) = 4.$$

2.
$$\int_1^2 x^3 = \frac{1}{4}x^4\bigg|_1^2 = \frac{1}{4}(2^4 - 1^4) = \frac{15}{4}.$$

3.
$$\int_0^{\pi/2} \cos(x) = \sin(x)\bigg|_0^{\pi/2} = \sin\left(\frac{\pi}{2}\right) - \sin(0) = 1.$$

4.
$$\int_0^{\pi/2} (\cos(x)+1) = (\sin(x)+x)\bigg|_0^{\pi/2} = \left(\sin\left(\frac{\pi}{2}\right) + \frac{\pi}{2}\right) - (\sin(0)+0) = 1 + \frac{\pi}{2}.$$

5.
$$\begin{aligned}\int_0^{\pi/2} (\cos(x) + \sin(x)) &= (\sin(x) - \cos(x))\bigg|_0^{\pi/2} \\ &= \left(\sin\left(\frac{\pi}{2}\right) - \cos\left(\frac{\pi}{2}\right)\right) - (\sin(0) - \cos(0)) \\ &= (1-0) - (0-1) = 2.\end{aligned}$$