Math 111 lecture for Friday, Week 10

Finding antiderivatives mean reversing the operation of taking derivatives. Today we'll consider reversing the chain rule and the product rule.

Substitution technique. Recall the chain rule:

$$(f(g(x)))' = f'(g(x))g'(x).$$

In terms of antiderivatives, this means

$$\int f'(g(x))g'(x)\,dx = f(g(x)) + c.$$

For example,

$$\int 10(x^3 + 4x + 2)^9(3x^2 + 4)\,dx = (x^3 + 4x + 2)^{10} + c.$$

Here, $f(x) = x^{10}$ and $g(x) = x^3 + 4x + 2$.

The technique of substitution is a formalism that helps in detecting the presence of the chain rule. Here's how it works. We know that

$$\int f'(g(x))g'(x)\,dx = f(g(x)) + c.$$

Define $u(x) = g(x)$. Then

$$\frac{du}{dx} = g'(x).$$

We abuse this notation by writing

$$du = g'(x)dx$$

and then substitute into the integral to get

$$\int f'(g(x))g'(x)\,dx = \int f'(u)du.$$

Then, by the FTC, we get

$$\int f'(u)\,du = f(u) + c.$$

Substituting back, using $u = g(x)$, we get

$$\int f'(g(x))g'(x)\,dx = \int f'(u)\,du = f(u) + c = f(g(x)) + c.$$

Example. Consider the indefinite integral

$$\int 3x^2(x^3+5)^6\,dx.$$

You may be able to immediately see how the chain rule applies. If not, as a general rule of thumb, look for a part of the integrand (the function you're integrating) that is "inside" another function and substitute. In this case, an obvious choice is to let

$$u = x^3 + 5.$$

Then using our notation from above,

$$du = 3x^2\,dx.$$

Substitute and integrate:

$$\int 3x^2(x^3+5)^6\,dx = \int (\underbrace{x^3+5}_{u})^6\,\underbrace{3x^2\,dx}_{du} = \int u^6\,du = \frac{1}{7}u^7 + c.$$

To get the final solution, substitute back:

$$\int 3x^2(x^3+5)^6\,dx = \frac{1}{7}(x^3+5)^7 + c.$$

Example. Integrate $\int x^4\cos(x^5)\,dx$. The "inside" function here is $u = x^5$. We get

$$du = 5x^4\,dx.$$

Therefore,

$$x^4\,dx = \frac{1}{5}du.$$

Now substitute:

$$\begin{aligned}\int x^4\cos(x^5)\,dx &= \int \cos(x^5)\,x^4\,dx \\ &= \int \frac{1}{5}\cos(u)\,du \\ &= \frac{1}{5}\sin(u) + c \\ &= \frac{1}{5}\sin(x^5) + c.\end{aligned}$$

Example. Here is a trickier example:

$$\int x\sqrt{1+5x}\,dx.$$

The inside function is $u = 1 + 5x$. So

$$du = 5\,dx \quad \Rightarrow \quad dx = \frac{1}{5}\,du.$$

We now need to substitute into the original integral to obtain an integral solely in the variable u—we need to get rid of all of the xs. Making a partial substitution in $x\sqrt{1+5x}\,dx$, we would get

$$x\sqrt{1+5x}\,dx = \frac{1}{5}\,x\sqrt{u}\,du,$$

but we need to get rid of the x remaining in this expression. Here's how: since $u = 1 + 5x$, we can solve for x in terms of u:

$$u = 1 + 5x \quad \Rightarrow \quad x = \frac{1}{5}(u-1).$$

Thus,

$$x\sqrt{1+5x}\,dx = \frac{1}{5}\,x\sqrt{u}\,du = \frac{1}{25}(u-1)\sqrt{u}\,du.$$

So

$$\begin{aligned}
\int x\sqrt{1+5x}\,dx &= \int \frac{1}{25}(u-1)\sqrt{u}\,du \\
&= \int \frac{1}{25}(u-1)u^{1/2}\,du \\
&= \frac{1}{25}\int (u-1)u^{1/2}\,du \\
&= \frac{1}{25}\int (u^{3/2} - u^{1/2})\,du \\
&= \frac{1}{25}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + c \\
&= \frac{1}{25}\left(\frac{2}{5}(1+5x)^{5/2} - \frac{2}{3}(1+5x)^{3/2}\right) + c.
\end{aligned}$$

WARNING. Be careful with limits of integration when using substitutions. For example, using the substitution $u = x^5 + 1$ and $du = 5x^4\,dx$, we get

$$\int_0^1 x^4(x^5+1)^6\,dx = \frac{1}{5}\int_1^2 u^6\,du = \frac{1}{35}u^7\Big|_1^2 = \frac{1}{35}(2^7 - 1^7) = \frac{127}{35}.$$

The limits of integration change after the substitute since $u = 1$ when $x = 0$ and $u = 2$ when $x = 1$.

As an alternative, you could first just compute the *indefinite* integral (using the same substitution):

$$\int x^4(x^5+1)^6\,dx = \frac{1}{5}\int u^6\,du = \frac{1}{35}u^7 = \frac{1}{35}(x^5+1)^7 + c.$$

Then use the FTC:

$$\int_0^1 x^4(x^5+1)^6\,dx = \frac{1}{35}(x^5+1)^7\Big|_0^1 = \frac{127}{35}.$$

Integration by parts. The integration technique called *integration by parts* originates from the product rule:

$$(uv)' = u'v + uv'.$$

Integrate:

$$\int (uv)' = \int (u'v + uv') = \int u'v + \int uv'.$$

Now, $\int (uv)'$ is the indefinite integral; so we must find a function whose derivative is $(uv)'$, but that's easy: uv. So

$$uv = \int u'v + \int uv'.$$

We now modify the notation to specify the argument of the function (the independent variable):

$$u(x)v(x) = \int u'(x)v(x)\,dx + \int u(x)v'(x)\,dx.$$

Using the notation $du = u'(x)dx$ and $dv = v'(x)dx$, we can write

$$uv = \int v\,du + \int u\,dv.$$

Rearranging, we get the form that is useful for integration:

$$\boxed{\int u\,dv = uv - \int v\,du.}$$

The utility of this formula is that it might be that $\int v\,du$ is an easier integral than $\int v\,du$.

Example. Compute $\int xe^x\,dx$? Note that since $(e^x)' = e^x$, it's trivial to integrate e^x: we have $\int e^x\,dx = e^x + c$. To integrate xe^x by parts, we need to choose u and dv appropriately. The following choice works:

$$\begin{aligned} u &= x \\ dv &= e^x\,dx. \end{aligned}$$

We then need to find du and v:

$$\begin{aligned} u &= x & du &= dx \\ dv &= e^x\,dx & v &= e^x. \end{aligned}$$

Applying the boxed formula, above:

$$\begin{aligned} \int xe^x\,dx &= \int u\,dv \\ &= uv - \int v\,du \\ &= xe^x - \int e^x\,dx \\ &= xe^x - e^x + c. \end{aligned}$$

It is easy to check that the solution is correct: differentiate $xe^x - e^x + c$, and you will get xe^x. (You'll need the product rule, naturally.)

We can then use the antiderivative we've found to compute definite integrals. For example,

$$\begin{aligned} \int_0^1 xe^x\,dx &= (xe^x - e^x)\Big|_0^1 \\ &= (1 \cdot e^1 - e^1) - (0 \cdot e^0 - e^0) \\ &= (e - e) - (0 - 1) \\ &= 1. \end{aligned}$$

Example. Compute $\int x\cos(x)\,dx$. By parts:

$$\begin{aligned} u &= x & du &= dx \\ dv &= \cos(x)\,dx & v &= \sin(x). \end{aligned}$$

Then

$$\begin{aligned} \int x\cos(x)\,dx &= \int u\,dv \\ &= uv - \int v\,du \\ &= x\sin(x) - \int \sin(x)\,dx \\ &= x\sin(x) + \cos(x) + c. \end{aligned}$$

Check:

$$\begin{aligned} (x\sin(x) + \cos(x) + c)' &= (x\sin(x))' + \cos'(x) \\ &= (\sin(x) + x\cos(x)) - \sin(x) \\ &= x\cos(x). \end{aligned}$$

Challenge. Compute $\int e^x\cos(x)\,dx$ by parts.