Math 111 lecture for Wednesday, Week 9

We start by reviewing the definition of the integral. Please see the previous lecture. The key words and notation you should know:

- Partition of a close interval $[a, b]$:
$$P = \{t_0, \dots, t_n\}$$
with
$$a = t_0 < t_1 < \cdots < t_n.$$

- The subintervals of the partition P:
$$[t_0, t_1], [t_1, t_2], \dots, [t_{n-1}, t_n].$$
The i-th subinterval is $[t_{i-1}, t_i]$. It's length is $t_i - t_{i-1}$. You should think of each of these as a base for a rectangle.

- The y-values for f on the i-th interval:
$$f([t_{i-1}, t_i]).$$
This is the set of heights of the graph of the function sitting over the interval $[t_{i-1}, t_i]$. Think of these as the possible heights for approximating rectangles with base $[t_{i-1}, t_i]$.

- $$M_i = \operatorname{lub} f([t_{i-1}, t_i]) \quad \text{and} \quad m_i = \operatorname{glb} f([t_{i-1}, t_i]).$$
These are the heights for the best over-estimating rectangle and under-estimating rectangle, respectively.

- Upper sum and lower sum for f with respect to P:
$$U(f, p) = M_1(t_1 - t_0) + M_2(t_2 - t_1) + \cdots + M_n(t_n - t_{n-1}) = \sum_{i=1}^{n} M_i(t_i - t_{i-1})$$
$$L(f, P) = m_1(t_1 - t_0) + m_2(t_2 - t_1) + \cdots + m_n(t_n - t_{n-1}) = \sum_{i=1}^{n} m_i(t_i - t_{i-1})$$
These are over- and under-estimates for the integral.

- Upper and lower integrals:

$$U\int_a^b f := \text{glb}\,\{U(f,P) : P \text{ a partition of } [a,b]\}$$

$$L\int_a^b f := \text{lub}\,\{L(f,P) : P \text{ a partition of } [a,b]\}\,.$$

  Recall that for each partition P, we get an over-estimate of the integral: $U(f,P)$. As P varies over all possible partitions, we get a whole set of over-estimates. We want to take the smallest of these. The problem, it is usually the case that the set of all over-estimates has no least element, just as the set $(3,8]$ has no least element. So we need to take the greatest lower bound. Similar comments apply to the lower integral.

- If $U\int_a^b f = L\int_a^b f$, the f is integrable and

$$\int_a^b f := L\int_a^b f = U\int_a^b f.$$

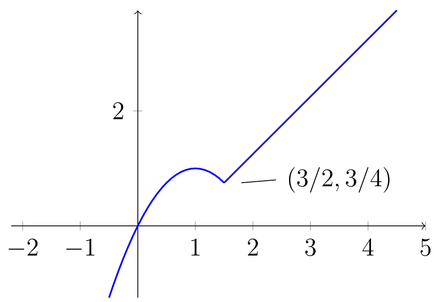
  If the integral exists, the set of upper sums is usually an interval of the form $(u,v]$ for some u and v, and the set of lower sums is an interval of the form $[w,u)$ for some w. The integral is the number u that's in between the two sets but not in either. We would have

$$L\int_a^b f = \text{lub}[w,u) = u = \text{glb}(u,v] = \int_a^b f.$$

Example. Here is an example that illustrates how to compute the upper and lower sums for a function with respect to a partition. The function will be

$$f(x) = \begin{cases} -(x-1)^2 + 1 & \text{if } x < \frac{3}{2}, \\ x - \frac{3}{4} & \text{if } x \geq \frac{3}{2}. \end{cases}$$

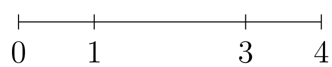
Here is the graph of f with the point where f is not differentiable marked:



We will estimate the area under the graph of f from $x = 0$ to $x = 4$, i.e., the integral $\int_0^4 f$ by computing upper and lower sums for f with respect to a partition of the interval $[0, 4]$. Take the partition to be

$$P = \{0, 1, 3, 4\}.$$

Here is a picture of P:



The subintervals of P are

$$[0, 1], \quad [1, 3], \quad \text{and} \quad [3, 4].$$

To computer the upper sum, $U(f, P)$, we first compute the least upper bounds of f on each subinterval:

$$M_1 = \operatorname{lub} f([0, 1]) = f(1) = 1$$
$$M_2 = \operatorname{lub} f([1, 3]) = f(3) = \frac{9}{4}$$
$$M_3 = \operatorname{lub} f([3, 4]) = f(4) = \frac{13}{4}.$$

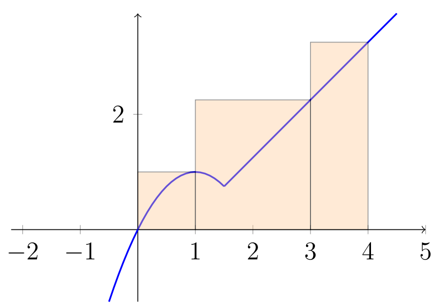
These values are the heights of our rectangles. The corresponding bases are the lengths of the three intervals:

$$1 - 0 = 1$$
$$3 - 1 = 2$$
$$4 - 3 = 1.$$

The upper sum for f for this partition is

$$\begin{aligned} U(f,P) &= M_1 \cdot 1 + M_2 \cdot 2 + M_3 \cdot \\ &= 1 \cdot 1 + \frac{9}{4} \cdot 2 + \frac{13}{4} \cdot 1 \\ &= \frac{35}{4} = 8.75. \end{aligned}$$

The upper sum is the sum of the areas of the rectangles in the following picture:



$U(f,P)$

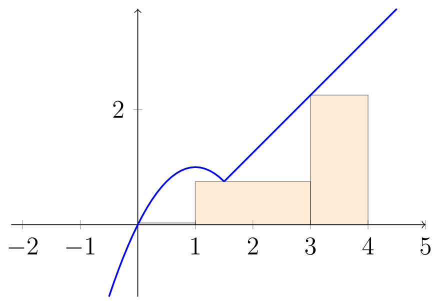
To find the lower sums, we repeat the above calculations but using the greatest lower bounds of f on the subintervals instead of the least upper bounds:

$$\begin{aligned} m_1 &= \operatorname{glb} f([0,1]) = f(0) = 0 \\ m_2 &= \operatorname{glb} f([1,3]) = f(1) = \frac{3}{4} \\ m_3 &= \operatorname{glb} f([3,4]) = f(3) = \frac{9}{4}. \end{aligned}$$

The bases of the corresponding rectangle have not changed. So the lower sum is

$$\begin{aligned} L(f,P) &= m_1 \cdot 1 + m_2 \cdot 2 + m_3 \cdot \\ &= 0 \cdot 1 + \frac{3}{4} \cdot 2 + \frac{9}{4} \cdot 1 \\ &= \frac{15}{4} = 3.75. \end{aligned}$$

The lower sum is the sum of the areas of the rectangles in the following picture:



$L(f, P)$

The first rectangle has height 0 and, hence, no area.
The actual area under f from $x = 0$ to $x = 4$ is $\int_0^4 f = \frac{49}{8} = 6.125$. We see

$$L(f, P) \leq \int_0^4 f \leq U(f, P)$$

since

$$3.75 \leq 6.125 \leq 8.75.$$

The percentage error for each sum (compared to the actual value) si

$$\frac{8.75 - 6.125}{6.125} \approx 42.9\%, \quad \frac{6.125 - 3.75}{6.125} \approx 38.6\%.$$

Not great. The lower sum is a bit better than the upper sum. One can see the error in the pictures for $U(f, P)$ and $L(f, P)$, above.