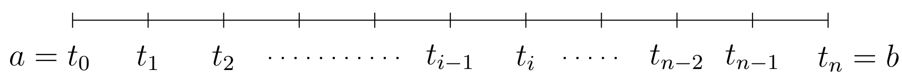
Math 111 lecture for Monday, Week 9

The integral. Now consider an arbitrary function f defined on an interval $[a, b]$. We would like to estimate the area under f by imitating what we just in the previous lecture with $f(x) = x/2$, above. Before, we divided the interval in question into n parts of equal length for convenience. In general, we allow division into *arbitrary length intervals*. To that end pick $n+1$ arbitrary points in the interval $[a, b]$, the first of which is a and the last of which is b:

$$a = t_0 < t_1 < t_2 < \cdots < t_n = b.$$

Here is a picture of the subdivision of $[a, b]$ into n parts (the dots connote an arbitrary number of tick marks):

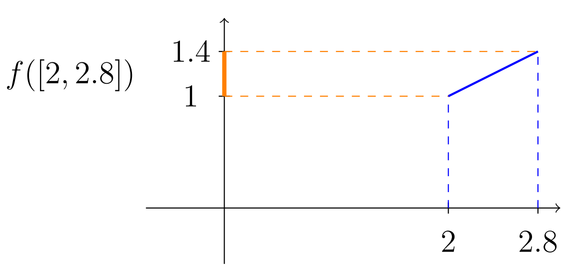


So we have divided the interval into *subintervals*. The first subinterval is $[t_0, t_1] = [a, t_1]$. The second subinterval is $[t_1, t_2]$, and so on. In general, the *i-th subinterval* is $[t_{i-1}, t_i]$.

The values the function f takes on the i-th interval is denoted $f([t_{i-1}, t_i])$:

$$f([t_{i-1}, t_i]) = \{f(x) : t_{i-1} \le x \le t_i\}.$$

This set is called the *image of* $[t_{i-1}, t_i]$ under f. For instance, if the function is $f(x) = x/2$ and the i-th interval is $[2, 2.8]$, then $f([2, 2.8]) = [1, 1.4]$. We can picture the image as the set of y-values of the function as x varies along the i-th interval:



The image of the interval $[2, 2.8]$ under $f(x) = \dfrac{x}{2}$.

To estimate the area under f we create rectangles based on each subinterval. To overestimate the area, we will take the height of each rectangle to be the maximum value of the function on its interval, and to underestimate the area, we will take the height to be the minumum value. We introduce notation for these heights:

$$M_i = \operatorname{lub} f([t_{i-1}, t_i]) \quad \text{and} \quad m_i = \operatorname{glb} f([t_{i-1}, t_i]).$$

So M_i is the least upper bound of all function values at points in the i-th subinterval, and m_i is the greatest lower bound. This means

$$m_i \leq f(x) \leq M_i$$

for all x satistfying $t_{i-1} \leq x \leq t_i$.

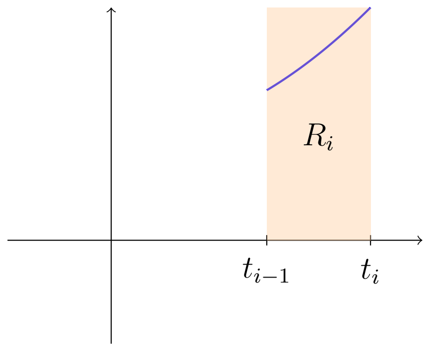
Time out for a technical point. Notice that we are using least upper bounds and greatest upper bounds instead of just taking the maximum value and the minimum value. There is a reason for that. Each subinterval is a closed bounded interval. If f is continuous, then the extreme value theorem guarantees that f has a maximum and a minimum value on the interval. However, if f is not continuous, it may not achieve its maximum or its minimum value on the interval. The set of function values will have a least upper bound and a greatest lower bound, however, as long as we assume the set of function values is bounded, which we will do from now on:

Assumption: From now on, we will assume that the set of values for f on the interval $[a, b]$ is bounded (both above and below), i.e.,

$$f([a, b]) = \{f(x) : a \leq x \leq b\},$$

is a bounded set of real numbers.

Back to defining the integral. We will next concentrate on creating an overestimate for the area under f. On the i-th subinterval, create a rectangle R_i with base $[t_{i-1}, t_i]$ and height M_i.



Rectangle on the i-th subinterval, overestimating the area.

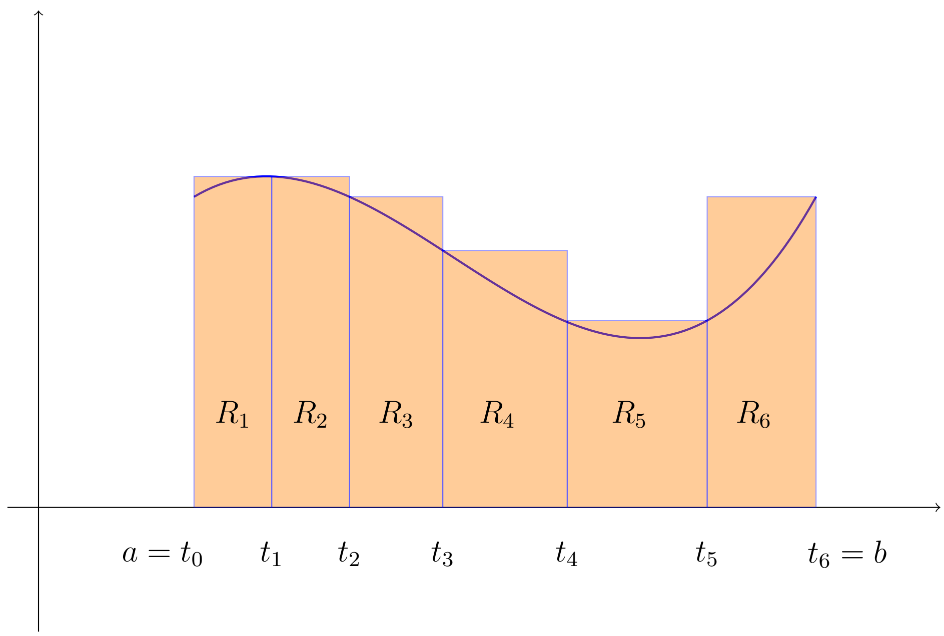
We have

$$\text{area}(R_i) = \text{height} \times \text{base} = M_i(t_i - t_{i-1}).$$

Adding up the areas of these rectangles gives an overestimate of the area called the *upper sum for f with respect to the partition* $P = \{t_0, t_1, \ldots, t_n\}$:

$$\begin{aligned} U(f,P) &:= \text{area}(R_1) + \text{area}(R_2) + \cdots + \text{area}(R_n) \\ &= M_1(t_1 - t_0) + M_2(t_2 - t_1) + \cdots + M_n(t_n - t_{n-1}) \\ &= \sum_{i=1}^{n} M_i(t_i - t_{i-1}). \end{aligned}$$

Below we illustrate the rectangles for an upper sum for some function f on a partition with 7 points (and, consequently, 6 subintervals):

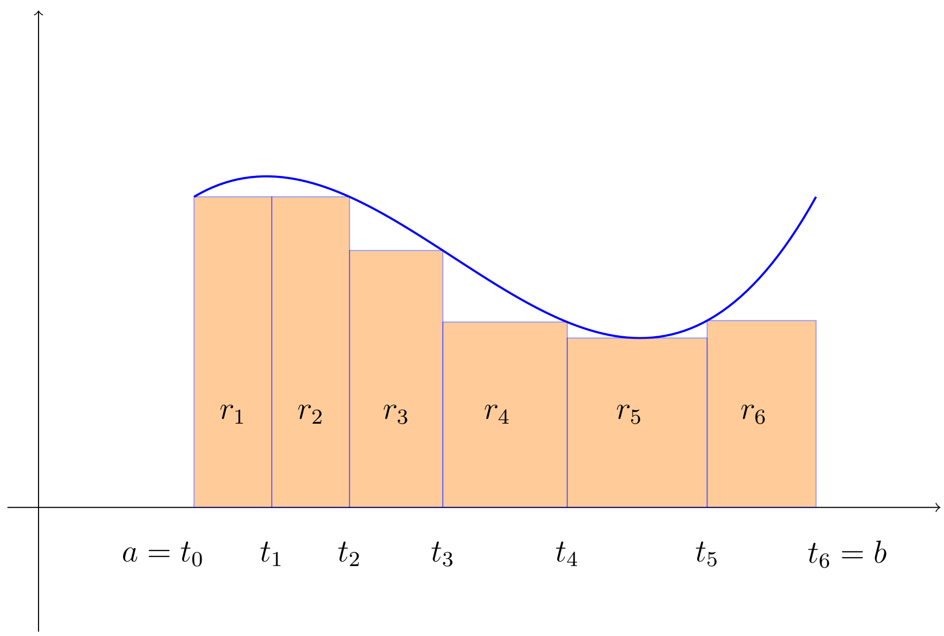


An upper sum $U(f, P)$ for some function f.

To get underestimates of the area, we repeat the above but now taking the heights of the rectangles to be the greatest lower bounds, m_i. Denote these rectangles by r_i. We define the *lower sum for f with respect to the partition P* to be the sum of the areas of these rectangles:

$$\begin{aligned} L(f,p) &:= \operatorname{area}(r_1) + \operatorname{area}(r_2) + \cdots + \operatorname{area}(r_n) \\ &= m_1(t_1 - t_0) + m_2(t_2 - t_1) + \cdots + m_n(t_n - t_{n-1}) \\ &= \sum_{i=1}^{n} m_i(t_i - t_{i-1}). \end{aligned}$$

The following picture shows the rectangles for the lower sum of some function f with respect to some partition P with 7 points (and 6 subintervals):



A lower sum $L(f, P)$ for some function f.

For each partition P of $[a, b]$, we get an upper sum—an overestimate of the area under f and a lower sum—an underestimate of the area. We first concentrate on the upper sums. Make a set consisting of all possible upper sums as we vary the partition:

$$\{U(f, P) : P \text{ a partition of } [a, b]\}\,.$$

This will, in general, be a set consisting of an infinite number of numbers, one for each of the infinite number of partitions P, and each an overestimate of the area we want. For any given choice of function f, it will almost certainly not have a smallest element. However, it turns out it does have a greatest lower bound (see Math 112). This is in some sense or "best overestimate". Officially, it is known as the *upper integral* for f:

$$U\int_a^b f := \text{glb}\,\{U(f, P) : P \text{ a partition of } [a, b]\}\,.$$

Similarly, we get one lower sum for each partition we choose, and we can consider the set of all possible underestimates of the area as the partition varies:

$$\{L(f,P) : P \text{ a partition of } [a,b]\}.$$

These are all underestimates, and in general there will be no greatest element in this set. However, it does have a least upper bound, and we define this to be the *lower integral* for f:

$$L\int_a^b f := \text{lub}\,\{L(f,P) : P \text{ a partition of } [a,b]\}.$$

So now we have a best overestimate and a best underestimate, and as might be expected (and proved in Math 112), we have

$$L\int_a^b f \leq U\int_a^b f.$$

For some unhappy functions, it turns out that these two numbers are not equal, and in that case we say f is *not integrable.* On the other hand, if they are equal, we say f is *integrable* on $[a,b]$, and in that case, the common value is *the integral* of f on $[a,b]$:

$$\int_a^b f := L\int_a^b f = U\int_a^b f.$$

Not every function is integrable, but we have the following:

Theorem. If f is a continuous function on the interval $[a,b]$, then it is integrable.

Proof. Math 112. $\square$

Summary of essential vocabulary. Let f be a function defined on a closed interval $[a,b]$.

1. The function f is *bounded* if the set of numbers $f(x)$ for $a \leq x \leq b$ is bounded above and below, i.e., $f([a,b])$ is bounded.

2. A *partition* P of $[a,b]$ is a finite set of points

   $$t_0 < t_1 < \cdots < t_n$$

   with $t_0 = a$ and $t_n = b$ and with n some positive integer. We write $P = \{t_0, t_1, \ldots, t_n\}$. For the following, fix such a partition P.

3. The *subintervals* for P are the intervals

$$[t_0, t_1], [t_1, t_2], \ldots, [t_{n-1}, t_n].$$

There are n subintervals in total. The i-th one is $[t_{i-1}, t_i]$. Each of these intervals is contained in the interval $[a, b]$. The *length* of the i-th subinterval is $t_i - t_{i-1}$.

4. We are going to estimate the area under the graph of f with rectangles based on each subinterval. Their heights are determined by the height of the graph of f:

$$\begin{aligned} M_i &:= \operatorname{lub} f([t_{i-1}, t_i]) \\ m_i &:= \operatorname{glb} f([t_{i-1}, t_i]). \end{aligned}$$

If f is continuous, these numbers are the maximum and minimum values of f on the i-th subinterval.

5. The partition P determines an overestimate and an underestimate for the area under the function f. These are called the *upper and lower sums for* f *with respect to* P:

$$\begin{aligned} U(f, p) &= \sum_{i=1}^{n} M_i(t_i - t_{i-1}) = M_1(t_1 - t_0) + M_2(t_2 - t_1) + \cdots + M_n(t_n - t_{n-1}) \\ L(f, p) &= \sum_{i=1}^{n} m_i(t_i - t_{i-1}) = m_1(t_1 - t_0) + m_2(t_2 - t_1) + \cdots + m_n(t_n - t_{n-1}). \end{aligned}$$

The i-th summand is the area of a rectangle whose base is the i-th subinterval.

6. For each partition P that we choose, we get an upper sum (overestimate) and a lower sum (underestimate) for the area under f. In an attempt to get the best over- and underestimates, we define the *upper and lower integrals* of f on $[a, b]$:

$$\begin{aligned} U\int_a^b f &:= \operatorname{glb}\{U(f, P) : P \text{ a partition of } [a, b]\} \\ L\int_a^b f &= \operatorname{lub}\{L(f, P) : P \text{ a partition of } [a, b]\}. \end{aligned}$$

7. Finally, if the upper and lower integrals are equal (as they are when f is continuous), we define the *integral* of f to be their common value:

$$\int_a^b f := L\int_a^b f = U\int_a^b f.$$