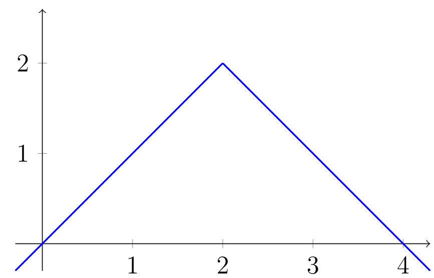
Math 111 lecture for Friday, Week 9

We will continue with examples designed to help understand the definition of the integral.

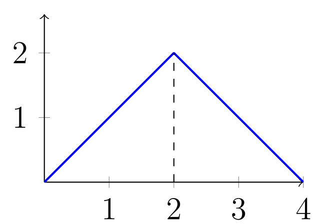
Example I. Consider the function $f(x) = 2 - |x - 2|$. Its graph appears below.



Exercises.

1. Use basic geometry to find the area under the graph of f on the interval $[0, 4]$.

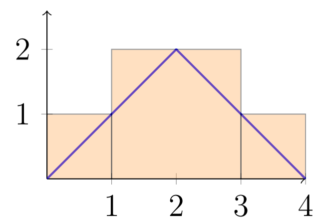
Solution. It's a triangle with base of length 4 and height 2. So the area is $\frac{1}{2} \cdot 4 \cdot 2 = 4$.



2. Let $P = \{0, 1, 3, 4\}$ be a partition of $[0, 4]$. Compute $U(f, P)$ and $L(f, P)$, the upper and lower sums for f on P.

Solution. For the upper sum, we get

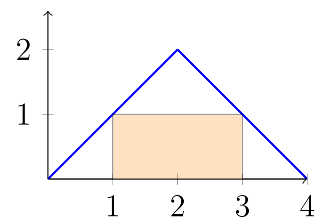
$$\begin{aligned} U(f, P) &= M_1(1 - 0) + M_2(3 - 1) + M_3(4 - 3) \\ &= 1 \cdot 1 + 2 \cdot 2 + 1 \cdot 1 \\ &= 6. \end{aligned}$$



Overestimate $U(f, P)$.

For the lower sum, we get

$$\begin{aligned} L(f, P) &= m_1(1-0) + m_2(3-1) + m_3(4-3) \\ &= 0 \cdot 1 + 1 \cdot 2 + 0 \cdot 1 \\ &= 2. \end{aligned}$$

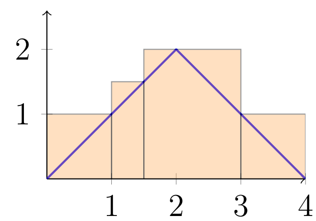


Underestimate $L(f, P)$.

3. Now consider the partition $Q = \left\{0, 1, \frac{3}{2}, 3, 4\right\}$ of $[0, 4]$. Compute $U(f, Q)$ and $L(f, Q)$. You should notice that the estimates are better.

Solution. For the upper sum, we get

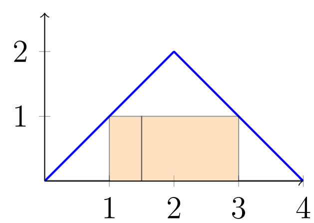
$$\begin{aligned} U(f, Q) &= M_1(1-0) + M_2(3/2-1) + M_3(3-3/2) + M_4(4-3) \\ &= 1 \cdot 1 + \frac{3}{2} \cdot \frac{1}{2} + 2 \cdot \frac{3}{2} + 1 \cdot 1 \\ &= 5\frac{3}{4} = \frac{23}{4} = 5.75. \end{aligned}$$



Overestimate $U(f, Q)$.

For the lower sum, we get

$$
\begin{aligned}
L(f, Q) &= m_1(1-0) + m_2(3/2-1) + m_3(3-3/2) + m_4(4-3) \\
&= 0 \cdot 1 + 1 \cdot \frac{1}{2} + 1 \cdot \frac{3}{2} + 0 \cdot 1 \\
&= 2.
\end{aligned}
$$



Underestimate $L(f, Q)$.

Here is what we have so far:

$$L(f, P) \leq L(f, Q) \leq \text{actual area } \leq U(f, Q) \leq U(f, P)$$

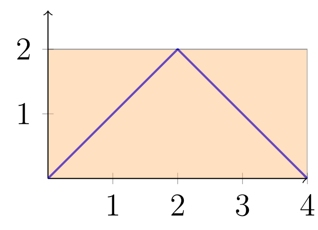
is

$$2 \leq 2 \leq 4 \leq 5.75 \leq 6.$$

4. Find a partition S of $[0, 4]$ for which $L(f, S)$ is smallest (the worst underestimate), and find a partition S' of $[0, 4]$ for which $U(f, S')$ is largest (the worst overestimate).

Solution. These will be given by the partition $S = S' = \{0, 4\}$ having only 2 points. For the upper sum, we get

$$U(f, S) = M_1(4 - 0) = 2 \cdot 4 = 8.$$



Overestimate $U(f,S)$.

For the lower sum, we get

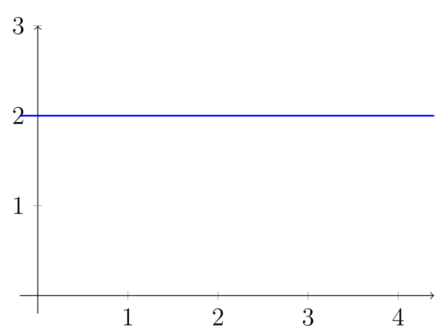
$$L(f,S) = m_1(4-0) = 0 \cdot 4 = 0.$$

5. Describe the set of real numbers that are possible upper sums as you vary the partitions. Do the same for lower sums.

Solution. The set of possible upper sums is $(4,8]$, and the set of possible lower sums is $[0,4)$. The point here is that the actual area, the integral, $\int_0^4 f(x)$, is sandwiched in between. We get the value of the integral as

$$L\int_0^4 f = \text{lub}[0,4) = 4 = \text{glb}(4,8] = U\int_0^4 f.$$

Example II. Consider the function the constant function $f(x) = 2$. Here is a graph:



Graph of $f(x) = 2$.

Use the definition of the integral to prove that $\int_1^4 f = 6$.

Solution. Consider the "coarsest partition"—the one containing only the endpoints of the interval

$$P = \{1, 4\}.$$

There is only one subinterval: the interval $[1, 4]$, itself. Its length is $4 - 1 = 3$. Since $f(x) = 2$ is a constant function, we have

$$M_1 = \operatorname{lub} f([1, 4]) = 2 \quad \text{and} \quad m_1 = \operatorname{glb} f([1, 4]) = 2.$$

It follows that

$$U(f, P) = L(f, P) = 2 \cdot (4 - 1) = 6.$$

Since $U(f, P)$ is *an* upper sum and $U\int_1^4 f$ is a *lower bound* for upper sums, we have

$$U\int_1^4 f \leq U(f, P).$$

Similarly, since $L(f, P)$ is *a* lower sum and $L\int_1^4 f$ is an *upper bound for the lower sums*, we have

$$L(f, P) \leq L\int_1^4 f.$$

Combining these inequalities, we get

$$\star\star \qquad L(f, P) \leq L\int_1^4 f \leq U\int_1^4 f \leq U(f, P). \qquad \star\star$$

This string of inequalities holds not just in this example but for all functions f and for all partitions P. However, for our particular f and the partition P that we have chosen, we get

$$6 = L(f, P) \leq L\int_1^4 f \leq U\int_1^4 f \leq U(f, P) = 6.$$

Since the number 6 appears at both ends, the upper and lower integrals must be equal to each other:

$$L\int_1^4 f = U\int_1^4 f = 6.$$

This proves that

$$\int_1^4 f = 6.$$

Example. It turns out that if a function is continuous or only has a finite number of discontinuities, then the function is integrable. So a non-integrable function is necessary hard to picture. Here's an example:

$$f\colon [0,1] \to [0,1]$$
$$x \mapsto \begin{cases} 1 & \text{if } x \text{ is irrational} \\ 0 & \text{if } x \text{ is rational.} \end{cases}$$

Between every two distinct numbers, there is both a rational and an irrational number. So this function is in fact not continuous as any point in $[0,1]$. Let $P = \{t_0, t_1, \ldots, t_n\}$ be any partition of $[0,1]$. Consider the i-th subinterval, $[t_{i-1}, t_i]$. This interval will contain both rational and irrational numbers. Hence, $M_i = 1$ and $m_i = 0$ for all i. This means that every approximating rectangle for the upper sum will have height 1, and thus, the upper sum will be

$$\begin{aligned} U(f,P) &= 1(t_1 - t_0) + 1(t_2 - t_1) + 1(t_3 - t_2) + 1(t_4 = t_3) + \cdots + 1(t_n - t_{n-1}) \\ &= -t_0 + t_n \\ &= 0 + 1 = 1. \end{aligned}$$

Notice how all of the inner t_i cancel! You should think about this geometrically, too: what would all of the rectangles look like? The lower sum will be

$$\begin{aligned} L(f,P) &= 0(t_1 - t_0) + 0(t_2 - t_1) + 0(t_3 - t_2) + \cdots + 0(t_n - t_{n-1}) \\ &= 0. \end{aligned}$$

So *every* upper sum is 1 and *every* lower sum is 0. This means that

$$U\!\int_a^b f := \operatorname{glb}\{U(f,P) : P \text{ a partition of } [a,b]\} = \operatorname{glb}\{1\} = 1,$$

and

$$L\!\int_a^b f := \operatorname{lub}\{L(f,P) : P \text{ a partition of } [a,b]\} = \operatorname{lub}\{0\} = 0.$$

It is easy to take the glb or lub of a set that contains only one element, as in these cases. So we have

$$L\int_a^b = 0 \lneq 1 = U\int_a^b f,$$

which means that f is not integrable. Now matter how we pick our partitions, the lower and upper sums are not going to get close to each other.