Math 111 lecture for Wednesday, Week 8

Our next goal is to define the integral of a function. If f is a nonnegative function defined on a closed bounded interval $[a, b]$, the integral $\int_a^b f$ will be used to define the area between the graph of f and the x-axis.

Upper and lower bounds. Let X be any subset of the real numbers, $\mathbb{R}$. An *upper bound* for X is *any* real number B that is at least as big as all the numbers in X, i.e.,

$$x \leq B \quad \text{for all } x \in X.$$

Similarly, a An *lower bound* for X is any real number b that is no greater than any numbers in X, i.e.,

$$b \leq x \quad \text{for all } x \in X.$$

Finally, the set X is simply called *bounded*, if it has both an upper and lower bound.

Examples.

1. Let $X = \{-2, 7, 9\}$, a set consisting of 3 real numbers. Then every number greater than or equal to 9 is an upper bound for X, and every number less than or equal to -2 is a lower bound. So 9, 9.1, and 27 are examples of upper bounds for X and -7, -7.234, and $-\pi$ are examples of lower bounds for X. The set X has upper and lower bounds, so X is bounded.

2. Let $Y = [3, 100) = \{x \in \mathbb{R} : 3 \leq x \leq 100\}$, a half-open, half-closed interval. The set Y is bounded: any number greater than or equal to 100 is an upper bound and any number less than or equal to 3 is a lower bound.

3. Let $Z = [0, \infty)$, the set of nonnegative real numbers. Then Z has no upper bounds, and every nonpositive number is a lower bound. The set Z is not bounded. It is only bounded below.

4. The set $\mathbb{R}$ of all real numbers has no upper bound and no lower bound. The same goes for the integers, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

Least upper bounds and greatest lower bounds. In general: if X has an upper bound, then it has infinitely many upper bounds, and if X has a lower bound, then it has infinitely many lower bounds. There is a property of the real numbers that say this: if a set X has an upper bound, then it has a *least upper bound*. The least upper bound, denoted $\text{lub}(X)$, it characterized by two properties: (i) it is an upper bound for X, and (ii) it is less than or equal to every upper bound for X.

Similarly, if X has a lower bound, then it has a *greatest lower bound*. It is denoted by $\text{glb}(X)$ had is characterized by the two properties, (i) is is a lower bound for X, and (ii) is is greater than or equal to every lower bound for X.

Thus, in some sense, $\text{lub}(X)$ and $\text{glb}(X)$ are the “best” upper and lower bounds, respectively, for X, provided they exist.

Example.

1. If $X = \{-2, 7, 9\}$, then $\text{lub}(X) = 9$ and $\text{glb}(X) = -2$.
2. If $Y = [3, 100)$, then $\text{lub}(X) = 100$ and $\text{glb}(X) = 3$.
3. If $Z = [0, \infty)$, then $\text{lub}(X)$ does not exist, and $\text{glb}(X) = 0$. Similarly, $\text{lub}((-\infty, 0]) = 0$ and $\text{glb}((-\infty, 0])$ does not exist.
4. The set $\mathbb{R}$ of all real numbers has neither a least upper bound nor a greatest lower bound. The same holds for the set of integers $\mathbb{Z}$.
5. Let
$$A = \left\{ \frac{1}{n} : n = 1, 2, 3, \dots \right\} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \right\}.$$
Then $\text{lub}(A) = 1$, and $\text{glb}(A) = 0$.
6. Let
$$B = \left\{ \frac{n}{n+1} : n = 1, 2, 3, \dots \right\} = \left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots \right\}.$$
Then $\text{lub}(B) = 1$, and $\text{glb}(A) = 1/2$.

NOTE WELL: Even if the least upper and greatest lower bounds for a set exist, they need not be in the set. For example, consider the set

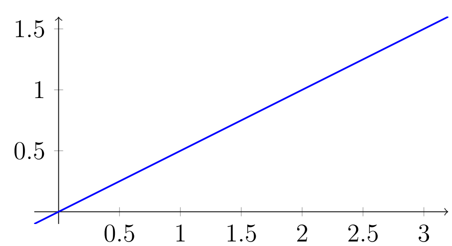
$$X = (0, 1) = \{x \in \mathbb{R} : 0 < x < 1\}.$$

Then $\mathrm{lub}(X) = 1$ and $\mathrm{glb}(X) = 0$. However, neither 0 nor 1 is in $(0,1)$. This interval, by definition does not contain its endpoints. It's hard to appreciate the importance of this fact now, but lub and glb are as important in defining the integral as limits were to defining the derivative. Recall that with derivatives, we need to find out what value the function of h

$$\frac{f(c+h) - f(c)}{h}$$

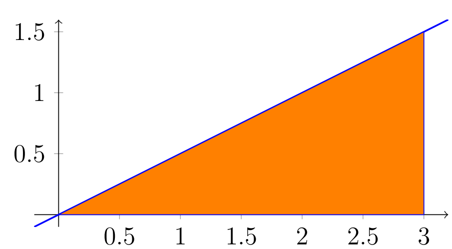
"should" have when h is 0 even though the function is actually undefined at that point. In a similar way, $\mathrm{lub}(X)$ is the number that should be the "biggest element of X" even though X may not have a biggest element. (For instance, the set $(0,1)$ has no largest element: 1 is not in $(0,1)$, and given any element in $x \in (0,1)$ there is another element in $y \in (0,1)$ such that $x < y$—just take y to be the number that is halfway between x and 1.)

Integration. We start off with a relatively simple example. Consider the function $f(x) = x/2$ on the interval $[0,3]$:



Graph of $f(x) = \dfrac{x}{2}$.

We are interested in computing the area of the colored region that is under the graph:



Graph of $f(x) = \dfrac{x}{2}$.

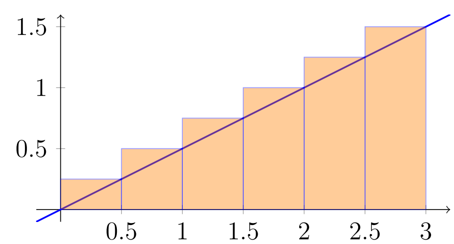
Since this is a triangle, its area is

$$\frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \times 3 \times \frac{3}{2} = \frac{9}{4}.$$

To motivate the definition of the integral, we are going to calculate this area in a more difficult way (but which has the advantage of generalizing to much more complicated functions).

The idea of integration is to compute area using only rectangles: give any region, divide the region up into rectangles as close as you can, then estimate the region by adding up the areas of the approximating rectangles. Of course, if the region is curved, you can never divide it up exactly into a finite number of rectangles, although, if you are willing to use very tiny rectangles, you can hope to estimate the area as close as you'd like. However, if you want the precise area, you can see that this rectangle method is going to lead to a limiting process of some sort.

Below, we create an overestimation of the area of the triangle:



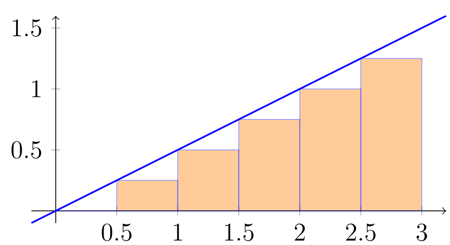
Graph of $f(x) = \dfrac{x}{2}$.

To estimate the area, we add the areas of these rectangles. The base of each rectangle is $1/2$, What about the height? Since the function is $f(x) = x/2$, to find the height, we evaluate f at the right-hand endpoint of the base of each rectangle. The result is

$$
\begin{aligned}
E &= \left(\frac{1}{2}\cdot\frac{1}{4}\right)+\left(\frac{1}{2}\cdot\frac{1}{2}\right)+\left(\frac{1}{2}\cdot\frac{3}{4}\right)+\left(\frac{1}{2}\cdot 1\right)+\left(\frac{1}{2}\cdot\frac{5}{4}\right)+\left(\frac{1}{2}\cdot\frac{3}{2}\right)\\
&= \frac{1}{2}\left(\frac{1}{4}+\frac{1}{2}+\frac{3}{4}+1+\frac{5}{4}+\frac{3}{2}\right)\\
&= \frac{1}{2}\left(\frac{1}{4}+\frac{2}{4}+\frac{3}{4}+\frac{4}{4}+\frac{5}{4}+\frac{6}{4}\right)\\
&= \frac{1}{2}\cdot\frac{1}{4}\left(1+2+3+4+5+6\right)\\
&= \frac{21}{8} = 2.625.
\end{aligned}
$$

This overestimates the actual area of the triangle, $9/4 = 2.25$.

We now create an underestimation of the area of the triangle:



Graph of $f(x) = \dfrac{x}{2}$.

The heights of the rectangles are now given by evaluating $f(x) = x/2$ at the left-hand endpoint of the base of each rectangle:

$$
\begin{aligned}
E' &= \left(\frac{1}{2} \cdot 0\right) + \left(\frac{1}{2} \cdot \frac{1}{4}\right) + \left(\frac{1}{2} \cdot \frac{1}{2}\right) + \left(\frac{1}{2} \cdot \frac{3}{4}\right) + \left(\frac{1}{2} \cdot 1\right) + \left(\frac{1}{2} \cdot \frac{5}{4}\right) \\
&= \frac{1}{2}\left(0 + \frac{1}{4} + \frac{1}{2} + \frac{3}{4} + 1 + \frac{5}{4}\right) \\
&= \frac{1}{2}\left(\frac{0}{4} + \frac{1}{4} + \frac{2}{4} + \frac{3}{4} + \frac{4}{4} + \frac{5}{4}\right) \\
&= \frac{1}{2} \cdot \frac{1}{4}\left(0 + 1 + 2 + 3 + 4 + 5\right) \\
&= \frac{15}{8} = 1.875.
\end{aligned}
$$

This underestimates the actual area of the triangle, $9/4 = 2.25$.

To get better over- and underestimates, we can divide up the base into more pieces and thus create better-fitting rectangles.