Math 111 lecture for Monday, Week 8

Curve sketching using derivatives

How can one approximate the graph of a function without using a computer? Here is a checklist:

1. Calculate where $f'(x) = 0$ and where $f'(x)$ does not exist (i.e., find the *critical points* of f), and evaluate the function at these points. Draw these points in your graph.

2. Determine the sign of f' between the critical points (in order to figure out how the slope of f changes).

3. Find the *zeros* of f, i.e., those points where $f(x) = 0$. Draw these points in your graph.

4. What happens to $f(x)$ as $x \to \infty$ and as $x \to -\infty$? (Are there horizontal asymptotes?)

5. Find places where the function "blows up", i.e., find any vertical asymptotes. If you find a vertical asymptote, how does f behave on either side of it?

6. To determine concavity, you can use the second derivative: $f''(x) > 0 \Rightarrow$ concave up, and $f''(x) < 0 \Rightarrow$ concave down.

Example 1. $f(x) = x^4 - 2x^2$.

1. Critical points:

$$\begin{aligned} f'(x) = 4x^3 - 4x = 0 \quad &\iff \quad x^3 = x \\ &\iff \quad x = 0 \quad \text{or} \quad x^2 = 1 \\ &\iff \quad x = -1, 0, 1. \end{aligned}$$

   We have $f(-1) = -1$, $f(0) = 0$, and $f(1) = -1$. So we plot the points $(-1, -1)$, $(0, 0)$, and $(1, -1)$.

2. The sign of f' between critical points:

slope of f:	down		up		down		up
f':	$-$		$+$		$-$		$+$
		-1		0		1	

3. Zeros of f:

$$f(x) = x^4 - 4x = 0 \quad \Longleftrightarrow \quad x = -\sqrt{2}, 0, \sqrt{2}.$$

4. Horizontal asymptotes

$$\lim_{x \to \infty} f(x) = \infty \quad \text{and} \quad \lim_{x \to -\infty} f(x) = \infty.$$

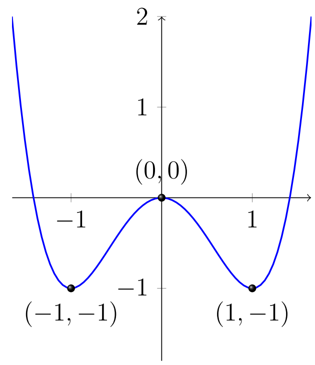
No horizontal asymptotes.

5. No vertical asymptotes.

6. Concavity:

$$\begin{aligned} f''(x) = 12x^2 - 4 > 0 \quad &\Longleftrightarrow \quad x^2 > \frac{1}{3} \\ &\Longleftrightarrow \quad x > \sqrt{\frac{1}{3}} \quad \text{or} \quad x < -\sqrt{\frac{1}{3}}. \end{aligned}$$

In particular, at the critical points, we have $f''(-1) = f''(1) = 8 > 0$ (concave up, hence, local minima), and $f(0) = -4 < 0$ (concave down, hence, a local maximum). We have $f''(x) = 0$ at $x = \pm\sqrt{1/3}$, and these are inflection points.



Graph of $f(x) = x^4 - 2x^2$.

Example 2. $f(x) = x^4 - 4x^3 = x^3(x-4)$.

1. Critical points:

$$\begin{aligned} f'(x) = 4x^3 - 12x^2 = 0 \quad &\iff \quad x^3 = 3x^2 \\ &\iff \quad x = 0, 3 \end{aligned}$$

   We have $f(0) = 0$, and $f(3) = -27$. So we plot the points $(0, 0)$, and $(3, -27)$.

2. The sign of f' between critical points:

slope of f:	down		down		up
f':	$-$		$-$		$+$
		0		3	

3. Zeros of f:

$$f(x) = x^4 - 4x^3 = 0 \quad \iff \quad x = 0, 4.$$

4. Horizontal asymptotes

$$\lim_{x\to\infty} f(x) = \infty \quad \text{and} \quad \lim_{x\to-\infty} f(x) = \infty.$$

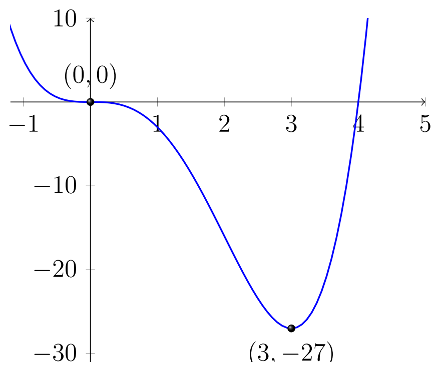
   No horizontal asymptotes.

5. No vertical asymptotes.

6. Concavity:

$$\begin{aligned} f''(x) = 12x^2 - 24x > 0 \quad &\iff \quad x^2 - 2x > 0 \\ &\iff \quad x(x-2) > 0 \end{aligned}$$

The points $x = 0$ and $x = 2$ are inflection points.



Graph of $f(x) = x^4 - 4x^3$.

Example 3. $f(x) = \dfrac{x^2 - 2x + 2}{x - 1}$.

1. Critical points:

   $$f'(x) = \frac{x(x-2)}{(x-1)^2} = 0 \quad \iff \quad x = 0, 2 \quad \text{and undefined at } x = 1.$$

   We have $f(0) = -2$, and $f(2) = 2$. So we plot the points $(0, -2)$, and $(2, 2)$.

2. The sign of f' between critical points:

slope of f:	up		down		down		up
f':	$+$		$-$		$-$		$+$
		0		1		2	

3. Zeros of f:

$$f(x) = \frac{x^2 - 2x + 2}{x - 1} = 0 \quad \Longrightarrow \quad x^2 - 2x + 2 = 0.$$

Using the quadratic equation, we see there are no real zeros:

$$x^2 - 2x + 2 = 0 \quad \Longrightarrow \quad x = \frac{2 \pm \sqrt{4 - 8}}{2}.$$

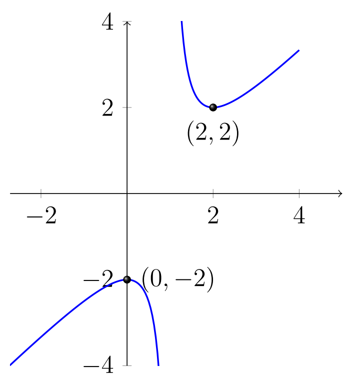
4. Horizontal asymptotes

$$\lim_{x \to \infty} f(x) = \infty \quad \text{and} \quad \lim_{x \to -\infty} f(x) = -\infty.$$

No horizontal asymptotes. However, note that for large x, we have $f(x) \approx \frac{x^2}{x} = x$. So the line $y = x$ is a kind of asymptote.

5. Vertical asymptote at $x = 1$.

6. Concavity: skipped since the second derivative is kind of messy.



Graph of $f(x) = \dfrac{x^2 - 2x + 2}{x - 1}$.