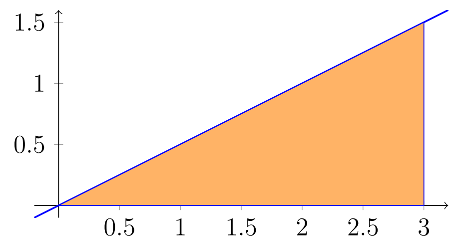
Math 111 lecture for Friday, Week 8

Warm-up to the definition of the derivative. Last time, we were considering a way of finding upper- and lower-bounds for the area under the graph of $f(x) = x/2$ from $x = 0$ to $x = 3$.



Graph of $f(x) = \frac{x}{2}$.

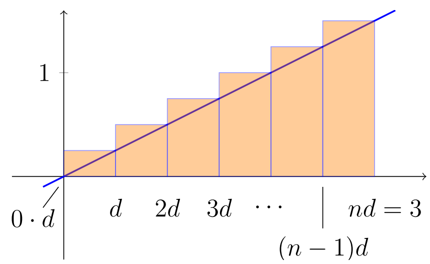
Of course, since the region is a triangle, we can easily compute its area with the usual formula: $(1/2) \times$ base $\times$ height $= 9/4 = 2.25$.

Last time, we tried approximating the area by dividing the interval $[0, 3]$ into 6 equal parts, then creating rectangles over each of these parts. The sum of the areas of the rectangles then gave an estimate of the area of the triangle. We want to generalize that approach now. Instead of dividing the region into 6 equal-sized pieces, let's divide it into n equal-sized pieces where n is any positive integer. Each piece would then have a length we will denote by d. So,

$$d := \frac{3-0}{n} = \frac{3}{n}.$$

The case $n = 6$ gives $d = 3/6 = 1/2$, as in our example.

The figure below attempts to illustrate the result of dividing the base of the triangle up into n parts (even though there are only 6 try to imagine a lot more parts signified by the dots, $\cdots$. (The term $(n-1)d$ appears below the other marks just because it wouldn't fit.)



Graph of $f(x) = \dfrac{x}{2}$.

Our task now is to add up the areas of the n rectangles (base$\times$ height). Each base has length $d = 3/n$. The rectangles have different heights. The height is determined by the right-hand endpoint of the base of the rectangle. Say that point is kd for some k. The rectangle's height is determined by the height of the graph, which would be $f(kd) = kd/2$. Hence, the area for this rectangle would be base$\times$height $= d \cdot kd/2$. Adding up the areas, we get

$$\text{sum of areas} = d \cdot \frac{d}{2} + d \cdot \frac{2d}{2} + d \cdot \frac{3d}{2} + \cdot + d \cdot \frac{(nd)}{2}$$

$$= \frac{d^2}{2} \cdot (1 + 2 + 3 + \cdots + n).$$

It turns out there is a useful closed formula for $1 + 2 + \cdots + n$. So we will pause in our calculation of the area to talk about that formula.

Lemma. For each $n > 0$,

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

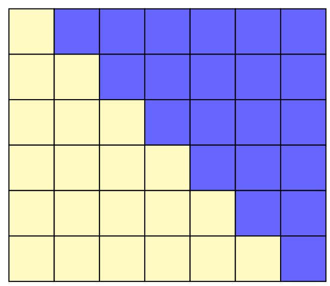
Proof. As a warm-up, we do the case $n = 6$. Here is the trick:

$$\begin{array}{rc} & 1+2+3+4+5+6 \\ + & 6+5+4+3+2+1 \\ \hline & 7+7+7+7+7+7 \end{array} \quad = 6 \cdot 7$$

Adding the sum twice gives $6 \cdot 7 = 42$. Divide by two to get the sum:

$$1 + 2 + 3 + 4 + 5 + 6 = \frac{6 \cdot 7}{2} = 21.$$

Here is a "proof by picture":



The 7×7 square contains our sum twice—once in yellow and once in blue. The proof clearly generalizes:

$$\begin{array}{c|ccccccccc|}
 & 1 & + & 2 & + & \cdots & + & (n-1) & + & n \\
+ & n & + & (n-1) & + & \cdots & + & 2 & + & 1 \\
\hline
 & (n+1) & + & (n+1) & + & \cdots & + & (n+1) & + & (n+1)
\end{array} \quad = n \cdot (n+1)$$

Divide by two to get the general sum formula:

$$1 + 2 + \cdots + (n-1) + n = \frac{n(n+1)}{2}.$$

$\square$

We now return to adding the areas of the rectangles:

$$\begin{aligned}
\text{sum of areas} &= d \cdot \frac{d}{2} + d \cdot \frac{2d}{2} + d \cdot \frac{3d}{2} + \cdot + d \cdot \frac{(nd)}{2} \\
&= \frac{d^2}{2} \cdot (1 + 2 + 3 + \cdots + n) \\
&= \frac{d^2}{2} \cdot \frac{n(n+1)}{2} \\
&= \frac{n(n+1)}{4} d^2 \\
&= \frac{n(n+1)}{4} \left(\frac{3}{n}\right)^2 \\
&= \frac{9}{4} \frac{(n+1)}{n} \\
&= \frac{9}{4} \left(1 + \frac{1}{n}\right).
\end{aligned}$$

For each n, we get an overestimate of the area of the triangle. Here is a set listing these overestimates for $n = 1, 2, 3, \ldots$:

$$X = \left\{ \left(\frac{9}{4} \cdot \frac{2}{1}\right), \left(\frac{9}{4} \cdot \frac{3}{2}\right), \left(\frac{9}{4} \cdot \frac{4}{3}\right), \ldots \right\}.$$

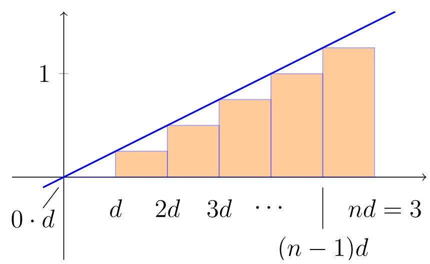
Notice that as n gets larger, the estimate gets smaller, each getting closer to the actually area, $9/4$, but never reaching it. The number we are looking for, the actual area, is the greatest lower bound of X:

$$\text{area of triangle} = \text{glb}(X).$$

Note however that the argument we have given to this point does not assure us that the area is $9/4$. We just know that the area is bounded above by $9/4$. So we should really write:

$$\text{area of triangle} \leq \text{glb}(X) = \frac{9}{4}.$$

In order to be sure that the area is 9/4, we need to look at underestimates, too. So we now repeat the above argument, but this time with underestimates for the area:



Graph of $f(x) = \dfrac{x}{2}$.

This time the rectangles have heights determined by the left-hand endpoints of the subintervals:

$$
\begin{aligned}
\text{sum of areas} &= d \cdot \frac{0 \cdot d}{2} + d \cdot \frac{1 \cdot d}{2} + d \cdot \frac{2d}{2} + \cdot + d \cdot \frac{(n-1)d}{2} \\
&= \frac{d^2}{2} \cdot (0 + 1 + 2 + \cdots + (n-1)) \\
&= \frac{d^2}{2} \cdot \frac{(n-1)n}{2} \\
&= \frac{(n-1)n}{4} d^2 \\
&= \frac{(n-1)n}{4} \left(\frac{3}{n}\right)^2 \\
&= \frac{9}{4} \frac{(n-1)}{n} \\
&= \frac{9}{4} \left(1 - \frac{1}{n}\right).
\end{aligned}
$$

For each n, we get an underestimate of the area of the triangle. A set listing these underestimates for $n = 1, 2, 3, \ldots$:

$$Y = \left\{ \left(\frac{9}{4} \cdot \frac{0}{1} \right), \left(\frac{9}{4} \cdot \frac{1}{2} \right), \left(\frac{9}{4} \cdot \frac{2}{3} \right), \ldots \right\}.$$

As n gets larger, this time the estimate gets larger, each getting closer to the actually area, $9/4$, but never reaching it. We have shown that

$$\frac{9}{4} = \text{glb}(Y) \leq \text{area of triangle}.$$

Combining this with our early calculation involving overestimates of the area, we get

$$\frac{9}{4} = \text{glb}(Y) \leq \text{area of triangle} \leq \text{lub}(X) = \frac{9}{4}.$$

This finally proves that the area of the triangle is $9/4$.