Math 111 lecture for Monday, Week 7

Optimization and related rates examples

Recall the general method for optimization problems. We have two main theorems:

Theorem 1. If f is differentiable at c and f has a local minimum or maximum at c, then $f'(c) = 0$.

Theorem 2. (The extreme value theorem, EVT) If f is continuous on a closed bounded interval $[a, b]$, then f has a (global) minimum and maximum on that interval.

These theorems suggest the following method for finding the minima and maxima of a functions:

Procedure for optimization. Suppose that f is a *continuous* function on a *closed bounded* interval $[a, b]$. Then the (global) minima and maxima for f occur among the following points:

(i) The points in (a, b) at which the derivative of f is 0.

(ii) The points in (a, b) at which f is not differentiable.

(iii) The endpoints, a and b.

Example. Let $f(x) = x^3 - x$. Find the minima and maxima for f on the interval $[-1, 2]$.

Solution. Follow the procedure outlined above. First collect the points:

(i) The function f is differentiable at all points in $[-1, 2]$ (in fact, f is differentiable at all points in $\mathbb{R}$. So this step gives us no interesting points to check.

(ii) We have

$$f'(x) = 3x^2 - 1 = 0 \quad \text{if and only if} \quad x = \pm\sqrt{\frac{1}{3}}.$$

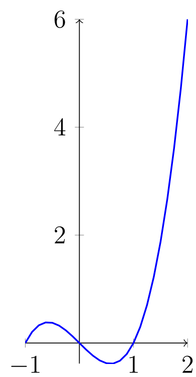
(iii) The endpoints of the interval are -1 and 2.

So we look for the extrema of f among the points $\pm\sqrt{1/3}$, -1, and 2. Evaluate f at these points:

$$f\left(-\sqrt{\frac{1}{3}}\right) = \frac{2}{3}\sqrt{\frac{1}{3}} \approx 0.38, \qquad f\left(\sqrt{\frac{1}{3}}\right) = -\frac{2}{3}\sqrt{\frac{1}{3}} \approx -0.38,$$

$$f(-1) = 0, \qquad f(2) = 6.$$

Thus, on the interval $[-1, 2]$, the minimum of f is $-2\sqrt{1/3}/3$, occurring at $\sqrt{1/3}$, and the maximum is 6, occurring at the endpoint, 2.



Graph of $f(x) = x^3 - x$ on the interval $[-1, 2]$.

While we have this picture, let's check that the derivative actually is giving the slope. We computed

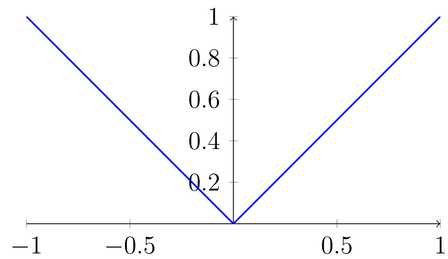
$$f'(x) = 3x^2 - 1.$$

Therefore, the graph should be sloped upwards if and only if $f'(x) > 0$. Compute:

$$f'(x) > 0 \quad \Leftrightarrow \quad 3x^2 - 1 > 0 \quad \Leftrightarrow \quad x^2 > \frac{1}{3} \quad \Leftrightarrow \quad x > \sqrt{\frac{1}{3}} \quad \text{or} \quad x < -\sqrt{\frac{1}{3}}.$$

This result is consistent with the graph, drawn above.

Example. This example illustrates the fact that in our procedure, we need to check points at which the derivative does not exist. Let $f(x) = |x|$ on the interval $[-1, 1]$:



Graph of $f(x) = |x|$ on the interval $[-1, 1]$.

Apply our procedure. The function f is continuous on a closed bounded interval. It is differentiable in the interior of the interval, $(-1, 1)$, except at the point $x = 0$. To see that it is not differentiable at 0, not that

$$\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{h - 0}{h} = 1$$

$$\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} \frac{(-h) - 0}{h} = -1.$$

This is because, $|h| = h$ when $h > 0$ and $|h| = -h$ when $h < 0$. Since the left-hand and right-hand limits are not equal, we know that the limit

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

does not exist. So f is not differentiable at 0.

Back to our procedure. The only critical point is $x = 0$, where the derivative does not exist. (At the other points in $(-1, 1)$, the derivative is ± 1, hence, nonzero.) So we need to check $x = 0$ and the endpoints $x = \pm 1$. We find that f has a minimum of 0, at $x = 0$, and a maximum 1, at $x = \pm 1$.

The next example illustrates what one might do if the extreme value theorem does not apply.

Example. Consider the function

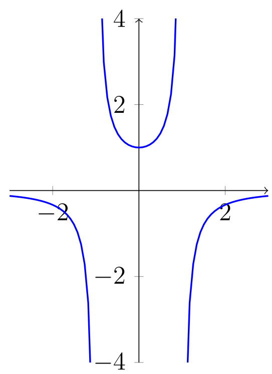
$$f(x) = \frac{1}{1 - x^2}$$

on the interval $(-1, 1)$. First note that f is continuous on $(-1, 1)$. However, the extreme value theorem does not apply, unfortunately, since $(-1, 1)$ is not a closed interval. So we need to be creative. First, it's clear the function blows up where $x = \pm 1$. In fact, as $x \to -1$ or $x \to 1$ from points inside the interval, the function f takes off to $+\infty$. Therefore, f has no maximum value on $(-1, 1)$. We can look for a minimum, though. Since f is differentiable on $(-1, 1)$, we know that f' will be 0 at any minimum (it needs to flatten out at these points). Computing the derivative of f using the quotient rule gives

$$f'(x) = \frac{2x}{(1 - x^2)^2}.$$

We have $f'(x) = 0$ only at the point $x = 0$.

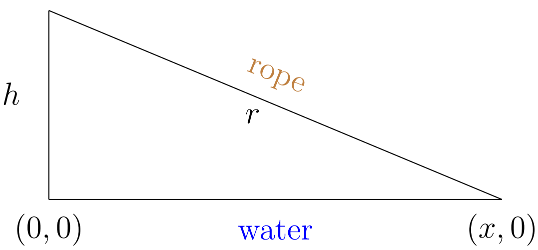
So far, we can only conclude that f has a *local* minimum at $x = 0$. However, note that for $x \in (-1, 1)$ we have that $f'(x) < 0$ for $x < 0$ and $f'(x) > 0$ for $x > 0$. This means that f is sloped downwards for $x < 0$ and sloped upwards for $x > 0$. This guarantees that $x = 0$ is a global minimum on the interval $(-1, 1)$.



Graph of $f(x) = \frac{1}{1-x^2}$.

One more related rates problem. Suppose you are standing on a dock pulling in a boat attached to a rope. If you pull the rope in at a constant rate, how does the speed at which the boat approaches the dock change?

SOLUTION: The relevant picture is:



The height of the point where the rope is being pulled is h feet above the water. We will assume that h is constant. The boat is x feet away from the dock, and the rope has length r. We are given that

$$\frac{dr}{dt} = -k = \text{ constant.}$$

We take $k > 0$, so the minus sign tells us that the rope is getting shorter over time. We are interested in the rate of change of x, i.e., in dx/dt. The equation relating the variables is

$$x^2 + h^2 = r^2.$$

Take derivatives with respect to time, remembering that h is constant:

$$2x\frac{dx}{dt} = 2r\frac{dr}{dt},$$

or

$$x\frac{dx}{dt} = r\frac{dr}{dt}.$$

Since $dr/dt = -k$, we get

$$x\frac{dx}{dt} = -kr.$$

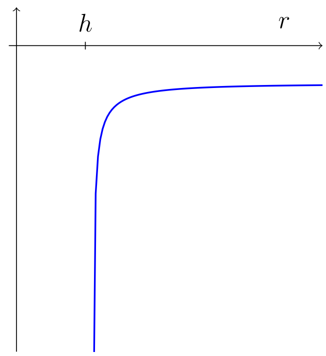
To get a solution in terms of r, we solve $x^2 + h^2 = r^2$ for x and substitute:

$$\frac{dx}{dt} = -k\frac{r}{\sqrt{r^2 - h^2}}.$$

So it is clear the speed of the boat is not constant. In fact, as time goes on, r approaches h, so the numerator is bounded around h with the denominator goes to 0, so

$$\lim_{r \to h} \frac{dx}{dt} = \lim_{r \to h} -k\frac{r}{\sqrt{r^2 - h^2}} = -\infty.$$

The graph of dx/dt as a function of r for $r > h$ looks something like this:



Graph of $\dfrac{dx}{dt}$ as a function of r.

In reality, it would be impossible to keep pulling in the rope at a constant speed.