Math 111 lecture for Friday, Week 7

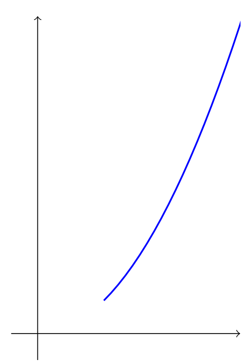
Curve sketching and the second derivative test

Second derivatives. If f is differentiable, then its derivative $f'(x)$ is a function of x and it makes sense to take its derivative. The derivative of $f'(x)$ is called the *second derivative of* f and denoted $f''(x)$.

Example. If $f(x) = x^4 - 3x^2 + 2$, then $f'(x) = 4x^3 - 6x$, and $f''(x) = 12x^2 - 6$.

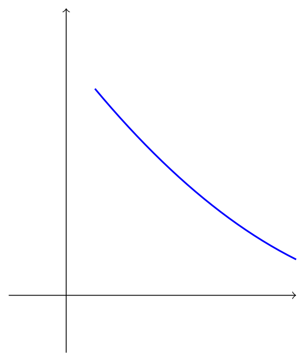
Just as $f'(x)$ gives the rate of change or slope of f, the second derivative, $f''(x)$, gives the rate of change or slope of $f'(x)$. If $f(t)$ gives the distance a particle has traveled after times t, the $f'(t)$ is the speed of the particle at time t, and $f''(t)$ is the acceleration of the particle at time t.

From the graph of f, we can tell where $f'(x) > 0$ by looking at the places where the slope of f is positive. Imagine what the graph of f looks like at places where $f''(x) > 0$. The slope of the graph would have to be increasing, which is called being *concave up*. That could happen at places where the graph is sloped upwards but getting steeper, as pictured below:



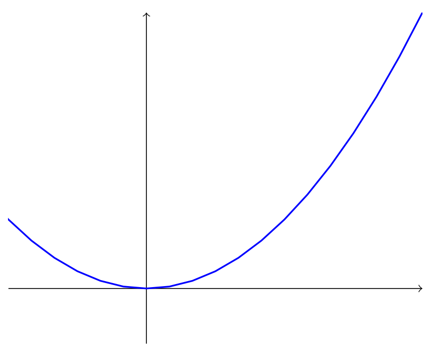
Graph of a function f with $f'' > 0$.

The slope of f can also be increasing if it goes from having a negative slope to having a less negative slope, as pictured below:



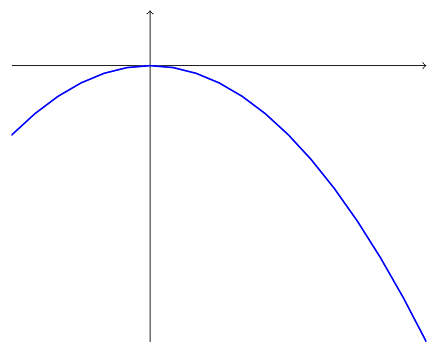
Graph of a function f with $f'' > 0$.

Another possibility is if the slope of f is changing from negative to positive, as pictured below:



Graph of a function f with $f'' > 0$.

Similarly, a graph with $f'' < 0$ is *concave down*:



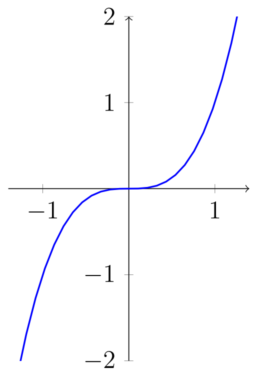
Graph of a function f with $f'' < 0$.

In summary,

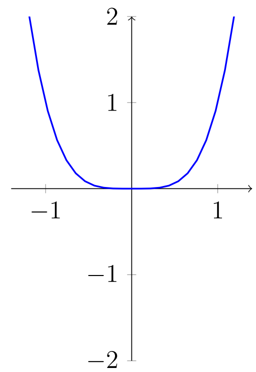
$$\begin{aligned} f''(x) > 0 &\implies \text{concave up} \\ f''(x) < 0 &\implies \text{concave down.} \end{aligned}$$

The second derivative test. Suppose that f is a function whose first and second derivatives exist, and say that $f'(c) = 0$. There are three possibilities:

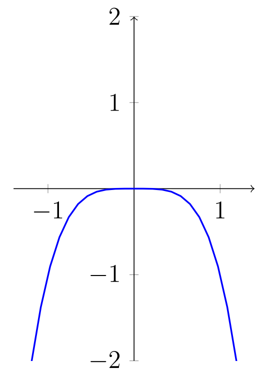
- $f'(c) = 0$ and $f''(c) > 0$. This means that f flattens out at c and is concave up around c. Therefore, c is a **local minimum** of f. (For example, consider $f(x) = x^2$ at $c = 0$.)
- $f'(c) = 0$ and $f''(c) < 0$. This means that f flattens out at c and is concave down around c. Therefore, c is a **local maximum** of f. (For example, consider $f(x) = -x^2$ at $c = 0$.)
- $f'(c) = 0$ and $f''(c) = 0$. This case is more complicated. For example, each of the following functions has $f'(0) = f''(0) = 0$:



$f(x) = x^3$.



$f(x) = x^4$.

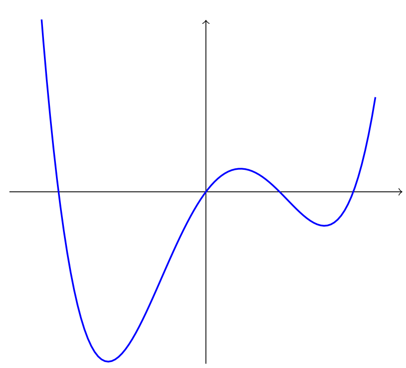


$f(x) = -x^4$.

There are three subcases:

- $f'(c) = 0 = f''(c) = 0$, and there is an interval about c on which $f'(x) < 0$ to the left of c and $f'(x) > 0$ to the right of c. Then f has a local minimum at c.
- $f'(c) = 0 = f''(c) = 0$, and there is an interval about c on which $f'(x) > 0$ to the left of c and $f'(x) < 0$ to the right of c. Then f has a local maximum at c.
- $f'(c) = 0 = f''(c) = 0$, and there is an interval about c on which f' has the same sign on either side of c. Then c is a *point of inflection* for f.

Exercise. For the function pictured below, what are the signs of $f'(x)$ and $f''(x)$ at each point x?



Exercise. For the function pictured below, what are the signs of $f'(x)$ and $f''(x)$ at each point x?