Math 111 lecture for Wednesday, Week 6

OPTIMIZATION

Intervals. An *interval* of the real numbers $\mathbb{R}$ is the set of all numbers between two given numbers. It may or may not contain the endpoints. The following are examples of intervals:

$$\begin{aligned}(-4,6) &= \{x \in \mathbb{R} : -4 < x < 6\}\\ (-4,6] &= \{x \in \mathbb{R} : -4 < x \leq 6\}\\ [-4,6) &= \{x \in \mathbb{R} : -4 \leq x < 6\}\\ [-4,6] &= \{x \in \mathbb{R} : -4 \leq x \leq 6\}.\end{aligned}$$

Intervals can also involve ∞, as in the following examples:

$$\begin{aligned}(-\infty,5) &= \{x \in \mathbb{R} : x < 5\}\\ (-\infty,5] &= \{x \in \mathbb{R} : x \leq 5\}\\ (5,\infty) &= \{x \in \mathbb{R} : x > 5\}\\ [5,\infty) &= \{x \in \mathbb{R} : x \geq 5\}\\ (-\infty,\infty) &= \mathbb{R}.\end{aligned}$$

The final example shows that the set of all real numbers, $\mathbb{R}$, is considered to be an interval.

Open, closed, and bounded intervals. An interval is *open* if it contains neither of its endpoints. Thus, and interval is open if it has one of the following forms

$$(a,b), \quad (-\infty,b), \quad (a,\infty), \quad (-\infty,\infty).$$

where a and b are real numbers. An interval is *closed* if it has one of the following forms:

$$[a,b], \quad (-\infty,b], \quad [a,\infty), \quad (-\infty,\infty).$$

Note that the set of all real number, $\mathbb{R} = (-\infty,\infty)$ is both open and closed. **Warning:** The notions of open and closed intervals are not opposites. For example, the interval $[2,6)$ is neither open nor closed.

An interval is *bounded above* if there exists a real number larger than all numbers in the interval. An interval is *bounded below* if there exists a real number smaller than all numbers in the interval. An interval is *bounded* if it is bounded above and below. Some examples:

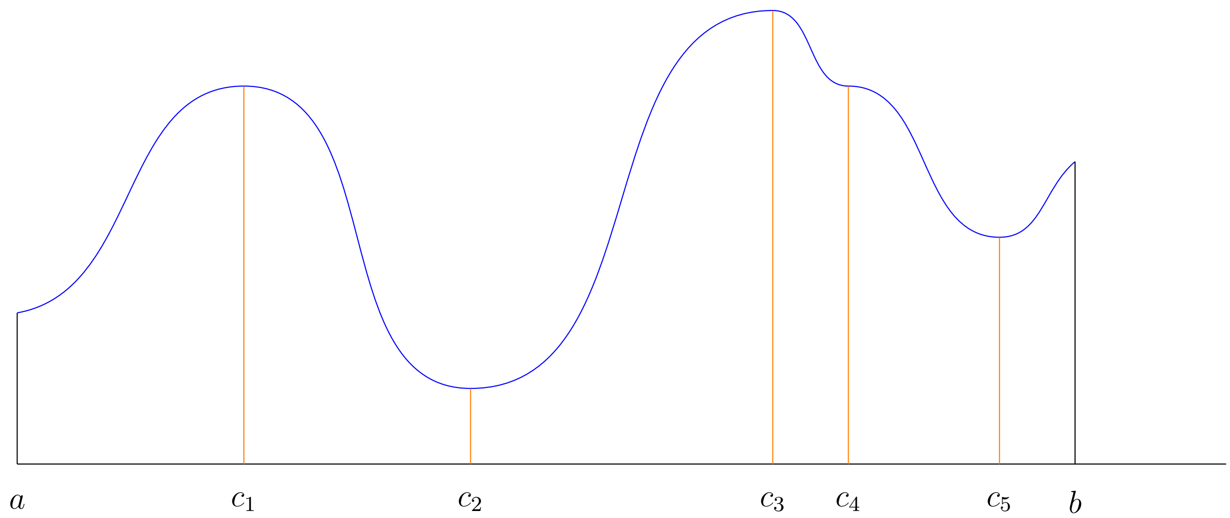
$$\begin{aligned}
\text{bounded above :} &\quad (-\infty, 3), \quad (-8, 5], \quad [0, 1] \\
\text{bounded below :} &\quad [3, \infty), \quad (-8, 5], \quad [0, 1] \\
\text{bounded :} &\quad (-8, 5], \quad [0, 1], \quad (0, 1).
\end{aligned}$$

Extrema. Let f be a function defined on an interval I, and let c be an element of I. Then

- f has a *minimum* at $c \in I$ if $f(c) \leq f(x)$ for all $x \in I$.
- f has a *maximum* at $c \in I$ if $f(c) \geq f(x)$ for all $x \in I$.

The minima and maxima are called *extrema*. Minima and maxima like these are sometimes called *global* minima and maxima to distinguish them from the following:

- f has a *relative (or local) minimum* at $c \in I$ if there exists an open interval containing c and contained in I on which f has a minimum at c.
- f has a *relative (or local) maximum* at $c \in I$ if the exists an open interval containing c and contained in I on which f has a maximum at c.



Graph of a function f.

c_1 - local maximum
c_2 - minimum (and local minimum)
c_3 - maximum (and local maximum)
c_4 - $f'(c_4) = 0$ but neither local minimum nor local maximum
c_5 - local minimum.

The endpoints a and b are not (global) minima or maxima, and they can't be local minima or maxima since there is no open interval containing these points inside the interval $[a, b]$.

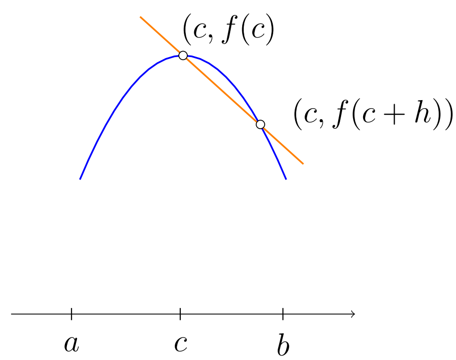
Warning. Here is a fine point: notice the less than or equal signs in the definition of minima and maxima rather than strict inequalities. This means. for instance, that for a constant function, *every* point is both a minimum and a maximum.

Two main results of optimization theory.

For the first result, note that the slope of the graph of a function at a local minimum or maximum must be 0. That's the content of the following theorem:

Theorem 1. If f is differentiable at c and f has a local minimum or maximum at c, then $f'(c) = 0$.

Proof. We will consider the case of a local maximum. The case of a local minimum is similar. So suppose f has a local maximum at c as pictured below:



We have $f(x) \geq f(c)$ for all x in the interval (a, b). Consider

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}.$$

If $h > 0$, then $f(c+h) \leq f(c)$, so

$$\frac{f(c+h) - f(c)}{h} \leq 0.$$

It follows that for the right-hand limit

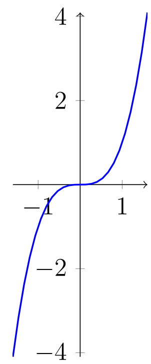
$$\lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \leq 0.$$

Reasoning similarly about the case where $h < 0$, we get for the left-hand limit

$$\lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} \geq 0.$$

(In this case, we'll have that both the numerator and denominator will be negative.) However, since $f'(c)$ exists, the right-hand and left-hand limits must be equal. So the only possibility is that they are both 0, which implies $f'(c) = 0$, as required. $\square$

Warning: The converse of the above theorem does not necessarily hold. A function can flatten out, i.e., it can have slope 0, at a point that is not a local minimum or maximum. For example, consider the point $x = 0$ for the function $f(x) = x^3$:



Graph of $f(x) = x^3$.

Theorem 2. (The extreme value theorem, EVT) If f is continuous on a closed bounded interval $[a, b]$, then f has a (global) minimum and maximum on that interval.

Proof. Math 112. $\square$

Note that the EVT assumes a closed interval. For instance, the function $f(x) = 1/x$ is continuous on the open interval $(0, 1)$ but is unbounded, hence has no maximum value. For that matter, the function $g(x) = x$ has no maximum or minimum on the open interval $(0, 1)$ even though the function is bounded on that interval. In contrast, the function $g(x)$ does have a maximum, 1, and a minimum, 0, on the *closed* interval $[0, 1]$.