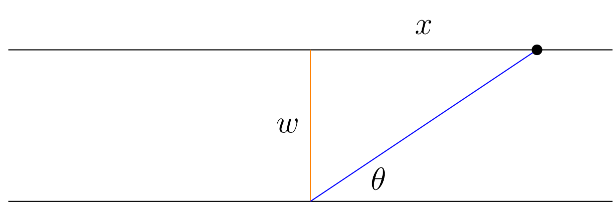
Math 111 lecture for Monday, Week 6

Related rates and implicit differentiation examples

Another related rates problem. Imagine a long hallway with paintings along the wall on one side. A slowly rotating surveillance camera is mounted on the opposite wall. How fast should its angle be changing in order for it to be scanning the opposite wall at a constant rate? Express your solution just in terms of the angle (which is what we would need to program the camera).

SOLUTION: Say the hall is some constant w feet wide. A picture of the situation is below:



We want the rate of change of x to be a constant. Say,

$$\frac{dx}{dt} = k.$$

The relation between θ and x is

$$\tan(\theta) = \frac{w}{x}. \tag{1}$$

Differentiating with respect to t gives, remembering that w is a constant and using the essential derivatives handout to find the derivative of the tangent function:

$$\begin{aligned} \frac{d}{dt}(\tan(\theta)) = \frac{d}{dt}\left(\frac{w}{x}\right) \quad &\Rightarrow \quad \frac{d}{dt}(\tan(\theta)) = w\frac{d}{dt}\left(\frac{1}{x}\right) \\ &\Rightarrow \quad \sec^2(\theta)\frac{d\theta}{dt} = -\frac{w}{x^2}\frac{dx}{dt} \\ &\Rightarrow \quad \sec^2(\theta)\frac{d\theta}{dt} = -\frac{kw}{x^2}. \end{aligned}$$

To express the solution solely in terms of θ, we can compute $1/x^2$ from equation (1):

$$\frac{1}{x^2} = \frac{\tan^2(\theta)}{w^2}.$$

Substitute this into the previous equation:

$$\sec^2(\theta)\frac{d\theta}{dt} = -\frac{kw}{x^2} = -kw \cdot \frac{\tan^2(\theta)}{w^2} = -k\frac{\tan^2(\theta)}{w}.$$

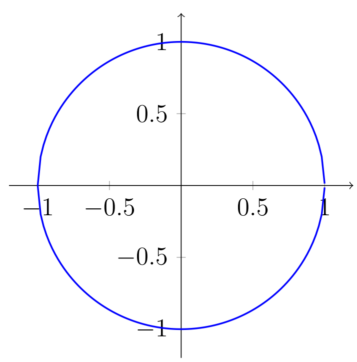
Solving for $d\theta/dt$:

$$\begin{aligned}\frac{d\theta}{dt} &= -k\frac{\tan^2(\theta)}{w} \cdot \frac{1}{\sec^2(\theta)} \\ &= -\frac{k}{w}\frac{\sin^2(\theta)}{\cos^2(\theta)} \cdot \cos^2(\theta) \\ &= -\frac{k}{w}\sin^2(\theta).\end{aligned}$$

Does this answer make sense? The negative sign is OK: as x increases, θ is decreasing. Next, note that as θ goes to 0, we have $\sin(\theta)$ going to 0, too. This makes sense: if the hallway were infinitely long, the camera would need to move more and more slowly to keep the point x moving at a constant rate.

Implicit differentiation example. The unit circle centered at the origin has defining equation

$$x^2 + y^2 = 1.$$



The unit circle, $x^2 + y^2 = 1$.

This curve is not the graph of a function since there are multiple y-values for some of the x-values. (The top semicircle is the graph of $y = \sqrt{1-x^2}$, and the bottom semicircle is the graph of $y = -\sqrt{1-x^2}$.) We can compute the slope implicitly, though, by taking derivatives of both sides of the equation with respect to x:

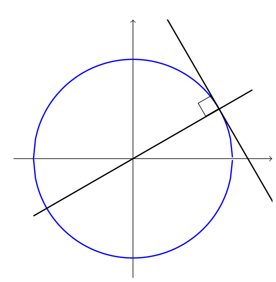
$$\begin{aligned} x^2 + y^2 = 1 \quad &\Rightarrow \quad \frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(1) \\ &\Rightarrow \quad 2x + 2y\frac{dy}{dx} = 0 \\ &\Rightarrow \quad \frac{dy}{dx} = -\frac{x}{y}. \end{aligned}$$

There is obviously trouble if $y = 0$. Note that the points on the circle where $y = 0$ are exactly the places where the tangent lines are vertical (having infinite slope). Note that $dy/dx = x/y = 0$ exactly where $x = 0$. These points occur at the top and bottom of the circle (the points $(0, \pm 1)$). For another example, consider the point $(\sqrt{2}/2, \sqrt{2}/2)$ on the unit circle. To find the slope at that point, plug into or equation for dy/dx:

$$\frac{dy}{dx} = -\frac{x}{y} = -\frac{\sqrt{2}/2}{\sqrt{2}/2} = -1.$$

Sure enough, the slope is -1, as expected.

Remark. Recall that if a line L has slope $m \neq 0$, then the lines perpendicular to L have slope $-1/m$, the negative reciprocal. Let (x, y) be a point on the unit circle with $x \neq 0$ and $y \neq 0$. The line L through the origin, $(0, 0)$, and (x, y) has slope y/x. Above, we computed that the slope of the tangent line at (x, y) is $-x/y$, the negative of reciprocal of L. So the tangent to the circle at (x, y) is perpendicular to the line through the origin and (x, y):



Implicit differentiation example. Consider the points (x, y) satisfying

$$\cos(xy) = \frac{1}{2}.$$

To compute the slope, use implicit differentiation (and the product rule):

$$\begin{aligned}
\cos(xy) = \frac{1}{2} \quad &\Rightarrow \quad \frac{d}{dx}\cos(xy) = \frac{d}{dx}\left(\frac{1}{2}\right) \\
&\Rightarrow \quad \frac{d}{dx}\cos(xy) = 0 \\
&\Rightarrow \quad -\sin(xy)\frac{d}{dx}(xy) = 0 \\
&\Rightarrow \quad -\sin(xy)\left(y + x\frac{dy}{dx}\right) = 0
\end{aligned}$$

As long as $\sin(xy) \neq 0$ and $y \neq 0$, this implies

$$\frac{dy}{dx} = -\frac{y}{x}.$$

For example, let's pick a point satisfying $\cos(xy) = 1/2$, such as $x = \pi/3$ and $y = 1$. The slope of the function at that point is

$$\frac{dy}{dx} = -\frac{3}{\pi} \approx -0.95.$$

The equation of the tangent line is $y = 1 - \frac{3}{\pi}(x - \frac{\pi}{3})$, which looks right: