Math 111 lecture for Friday, Week 6

OPTIMIZATION

Here are the two most basic theorems in optimization theory:

Theorem 1. If f is differentiable at c and f has a local minimum or maximum at c, then $f'(c) = 0$.

Proof. See the notes from last time. $\square$

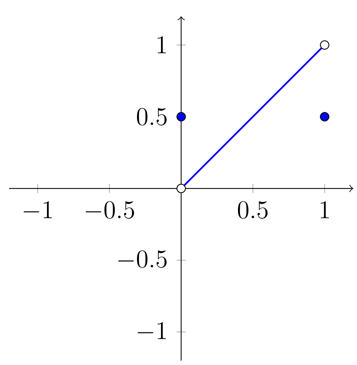
Theorem 2. (The extreme value theorem, EVT) If f is continuous on a closed bounded interval $[a, b]$, then f has a (global) minimum and maximum on that interval.

Proof. Math 112. $\square$

The extreme value theorem can fail if either f is *not continuous* or the interval is not *closed* and *bounded*. Some examples demonstrating this:

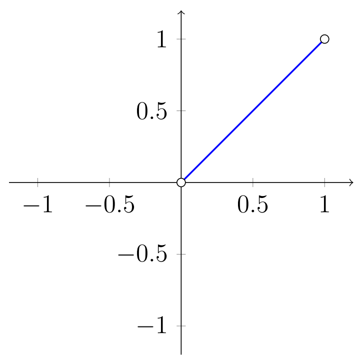
Example 1. Let $f\colon [0, 1] \to \mathbb{R}$. Here the interval is *closed* and *bounded*, but the function is *not continuous*. The function f has no minimum or maximum:

$$f(x) = \begin{cases} x & \text{if } 0 < x < 1 \\ \frac{1}{2} & \text{if } x = \pm 1. \end{cases}$$



Graph of f.

Example 2. Let $g\colon (0,1) \to \mathbb{R}$ be defined by $g(x) = x$. In this case, the function, g, is *continuous* and the interval is *bounded*. However, the interval is *not closed*. The function g has no minimum or maximum on that interval:



Graph of g.

Example 3. The function $h\colon (-\infty, \infty) \to \mathbb{R}$ defined by $h(x) = x$ is *continuous*. The interval on which h is defined is *closed*, but it is not *bounded*. The function h has no minimum or maximum.

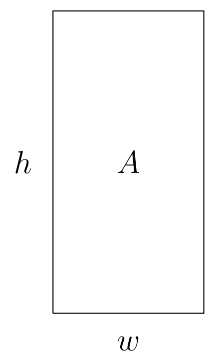
Example 4. One more example: the function $k(x) = 1/x$ is continuous on the open interval $(0,1)$, but has no maximum or minimum that interval. Theorem 2 does not apply since the interval is not closed.

Procedure for optimization. Given a continuous function f on a *closed bounded* interval $[a,b]$, to find the (global) minima and maxima for f:

1. Collect the points in (a,b) at which f is not differentiable.
2. Add the points at which the derivative of f is 0.
3. Add the endpoints, a and b.

The points satisfying 1 or 2 are called *critical points*. Evaluate f at the critical points and the endpoints. The smallest value will give the minimum and the largest will give the maximum.

Example 5. Suppose a farmer has ℓ feet of fence and wants to make a rectangular enclosure of the maximal area. What should the dimensions of the rectangle be? Similar to related rates problems, the first step in solving this problem is to draw a picture and label the relevant features:



We are given that the total length of fence is

$$\ell = 2w + 2h. \tag{1}$$

We are trying to find w and h in order to maximize the area

$$A = wh. \tag{2}$$

Since this is one-variable calculus, we need to write A as a function of one variable. We do this by solving for h in equation (1):

$$h = \frac{\ell}{2} - w.$$

Subsitute into equation (2):

$$A = \left(\frac{\ell}{2} - w\right) w = \frac{1}{2}\ell w - w^2. \tag{3}$$

We have now written A as a function of just one variable, w. Note that ℓ was fixed at the beginning of the problem—it's a constant.

Here is an important step that's easy to forget: to use our optimization technique, we should check the interval that constrains our variable, w. Ideally, it's a closed and bounded interval so that the EVT applies. In our case,

$$w \in [0, \ell/2],$$

and we see the relevant interval meets both conditions. So we can now apply our technique:

$$\frac{dA}{dw} = \frac{1}{2}\ell - 2w = 0 \quad \Longleftrightarrow \quad w = \frac{\ell}{4}.$$

So there is one critical point, $w = \ell/4$. Evaluate A at this point and at the endpoints of our interval using equation (3):

$$A\left(\frac{\ell}{4}\right) = \left(\frac{\ell}{2} - \frac{\ell}{4}\right) \cdot \frac{\ell}{4} = \frac{\ell^2}{16}$$

$$A(0) = A\left(\frac{\ell}{2}\right) = 0.$$

So the maximum area occurs when $w = \ell/4$. From equation (1) is then follows that $h = \ell/4$, too. So the way to maximize the area is to make a square.