Math 111 lecture for Wednesday, Week 5

Last time, we talked about the chain rule: if f and g are differentiable functions, then

$$(f(g(x))' = f'(g(x))g'(x).$$

Some examples of its application:

$$((3x^4 + 3x^2 + x + 1)^{25})' = 25(3x^4 + 3x^2 + x + 1)^{24}(12x^3 + 6x + 1)$$
$$(e^{\cos(x)})' = e^{\cos(x)}(-\sin(x)) = -\sin(x)e^{\cos(x)}.$$

It is common to see the chain rule expressed using a different notation which we will now describe. Suppose y is a function of x and x is a function of t. For instance, we could have

$$y = y(x) = x^3 \quad \text{and} \quad x = x(t) = t^2 + 4.$$

We can then express y as a function of t:

$$y = x^3 = (t^2 + 4)^3.$$

We can take the derivative of y, then, as a function of x or as a function of t. So simply writing y' for the derivative of y would be ambiguous. To make the distinction clear, we use the following notation:

$$\frac{dy}{dx} = 3x^2 \quad \text{and} \quad \frac{dy}{dt} = 3(t^2 + 4)^2(2t),$$

where we have used the chain rule in computing dy/dt. Also, note that $dx/dt = 2t$. In fact, using this notation, the chain rule can be expressed in this nice way:

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}.$$

Explicitly, with our example,

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = (3x^2) \cdot (2t) = (3(t^2 + 4)^2)(2t).$$

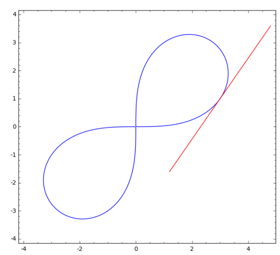
Note that as a trivial special case of the notation just introduced, suppose $x = t$. Then $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$ becomes

$$\frac{dy}{dx} = \frac{dy}{dx}\frac{dx}{dx} = \frac{dy}{dx} \cdot 1 = \frac{dy}{dx}.$$

For the rest of this lecture, we concentrate on a special application of the chain rule, using some of the notation just introduced. Suppose a function y is defined implicitly as a function of x by the equation

$$3(x^2+y^2)^2 = 100xy.$$

A picture of the points $(x, y) \in \mathbb{R}^2$ satisfying this equation appears below:



The picture show the tangent line to the curve at the point $(3, 1)$. Let's find the equation of this tangent line using the chain rule. The idea is to think of the equation as implicitly defining y as a function of x. Take the derivative of both sides of the equation with respect to x. For the right-hand side we get

$$\begin{aligned}\frac{d}{dx}(3(x^2+y^2)^2) &= 6(x^2+y^2)\left(\frac{d}{dx}(x^2+y^2)\right)\\ &= 6(x^2+y^2)\left(2x+2y\frac{dy}{dx}\right).\end{aligned}$$

for the left-hand side, we apply the product rule and the chain rule to get

$$\begin{aligned}\frac{d}{dx}(100xy) &= 100\left(\left(\frac{d}{dx}(x)\right)y + x\frac{dy}{dx}\right)\\ &= 100\left(1\cdot y + x\frac{dy}{dx}\right)\\ &= 100\left(y + x\frac{dy}{dx}\right)\end{aligned}$$

Setting the right-hand and left-hand sides equal, we get

$$6(x^2+y^2)\left(2x+2y\frac{dy}{dx}\right) = 100\left(y+x\frac{dy}{dx}\right).$$

We are interested in the point $(x, y) = (3, 1)$. Substituting $x = 3$ and $y = 1$ above:

$$60\left(6 + 2\frac{dy}{dx}\right) = 100\left(1 + 3\frac{dy}{dx}\right).$$

Solving for dy/dx now gives:

$$\frac{dy}{dx} = \frac{13}{9}.$$

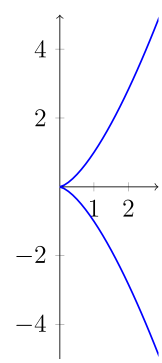
So the slope of the tangent line is $13/9$, and it passes through the point $(3, 1)$. So the tangent line has equation

$$y = 1 + \frac{13}{9}(x - 3).$$

Example. Here is one more example of computing the tangent line to an implicitly defined curve. Consider all of the point (x, y) that satisfy

$$y^2 = x^3.$$

A graph appears below:



The cuspidal cubic curve, $y^2 = x^3$.

The point $(1, 1)$ is on this curve since $1^3 = 1^2$. Let's compute an equation for the tangent line at that point. Taking the derivative of both sides of the defining equation with respect to x, we get

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(x^3) \Longrightarrow 2y\frac{dy}{dx} = 3x^2.$$

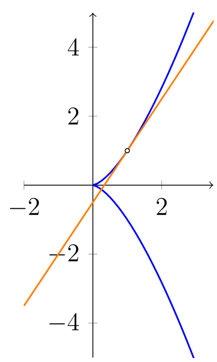
At the point $x = y = 1$, this becomes

$$2\frac{dy}{dx} = 3.$$

Hence, the slope at $(1, 1)$ is $3/2$. The line with slope $3/2$ and passing through the point $(1, 1,)$ has equation

$$y = 1 + \frac{3}{2}(x - 1).$$

Plotting this line with the curve gives the picture



The cuspidal cubic curve, $y^2 = x^3$.

In the case of this curve, it's easy to take the defining equation, $y^2 = x^3$, and solve for y. This yields

$$y = x^{3/2} \quad \text{or} \quad y = -x^{3/2}.$$

It's easy to see both branches of this function in the picture of the curve above. The part of the curve passing through $(1, 1)$ would be on the branch defined by $y = x^{3/2}$. We can now use ordinary methods to find the slope at $x = 1$:

$$y' = \frac{3}{2}x^{1/2}.$$

Evaluating at $x = 1$ gives $y'(1) = 3/2$. So the slope is $3/2$, as we found earlier.