Math 111 lecture for Monday, Week 5

THE CHAIN RULE

[Note: please take a look at the essential derivatives handout at our class homepage.]

Today, we'll do two things: the chain rule, and a trigonometry review.

Sometimes a function can be constructed from simpler functions using addition, multiplication, and division. In that situation the sum, product, and quotient rules for derivatives (our derivative theorem) gives us a means of computing the derivative of the function in terms of the derivatives of its simpler constituent functions. Another way to build a function is by composing simpler functions. The chain rule tells you how the derivative behaves in this situation.

Theorem. Suppose f and g are differentiable functions. Then

$$(f(g(x)))' = f'(g(x))g'(x).$$

Proof. Math 202 (where it is proved for functions of several variables—the case of one-variable calculus being a special case). □

Example. Compute the derivative of $(3x^4 + 2x^3 + 5x + 2)^{25}$. There are two ways to do this. The first is to expand out the 25-th power of a polynomial, then take the derivative of this polynomial. That would be quite painful. The second way is to use the chain rule by recognizing the given function as the composition of two simpler functions: we have

$$(3x^4 + 2x^3 + 5x + 2)^{25} = f(g(x))$$

where

$$f(x) = x^{25} \quad \text{and} \quad g(x) = 3x^4 + 2x^3 + 5x + 2.$$

To apply the chain rule, first take the derivatives of f and g:

$$f'(x) = 25x^{24} \quad \text{and} \quad g'(x) = 12x^3 + 6x^2 + 5.$$

Next, use the chain rule (you will need to compose f' and g):

$$\begin{aligned}((3x^4 + 2x^3 + 5x + 2)^{25})' &= (f(g(x)))' \\ &= f'(g(x))g'(x) \\ &= 25(3x^4 + 2x^3 + 5x + 2)^{24}(12x^3 + 6x^2 + 5).\end{aligned}$$

Example. We have seen in class that $(\sqrt{x})' = 1/(2\sqrt{x})$. We can use this to find the derivative of $\sqrt{3x^2+2x+2}$:

$$(\sqrt{3x^2+2x+2})' = \frac{1}{2\sqrt{3x^2+2x+2}} \cdot (6x+2) = \frac{3x+1}{\sqrt{3x^2+2x+2}}.$$

Here, $f(x) = \sqrt{x}$ and $g(x) = 3x^2+2x+2$.

Example. Given that $\sin'(x) = \cos(x)$ and $\cos'(x) = -\sin(x)$, the chain rule gives us:

(a)

$$(\cos^3(x))' = 3\cos^2(x)(\cos(x))' = 3\cos^2(x)(-\sin(x)) = -3\cos^2(x)\sin(x).$$

(Chain rule with $f(x) = x^3$, $g(x) = \cos(x)$.)

(b)

$$(\sin(\cos(x)))' = \cos(\cos(x))(\cos(x))' = -\cos(\cos(x))\sin(x).$$

(Chain rule with $f(x) = \sin(x)$, $g(x) = \cos(x)$.)

For multiple compositions, we can apply the chain rule multiple times:

$$(f(g(h(x))))' = f'(g(h(x))g'(h(x))h'(x).$$

Example.

(a)

$$\begin{aligned}(\sin((x^4+5x-2)^{10}))' &= \cos((x^4+5x-2)^{10})(10(x^4+5x-2)^9(4x^3+5)) \\ &= 10(x^4+5x-2)^9(4x^3+5)\cos((x^4+5x-2)^{10}).\end{aligned}$$

(b)

$$(\sin(\cos(x^3+5x)))' = -\cos(\cos(x^3+5x))\sin(x^3+5x)(3x^2+5).$$

Example. You can, of course, combine the chain rule with the sum, product, and quotient rules. Here, we use the fact that $(\ln(x))' = 1/x$.

$$\begin{aligned}(x^2\sin(\ln(x)))' &= (x^2)'\sin(\ln(x)) + x^2(\sin(\ln(x)))' \\ &= 2x\sin(\ln(x)) + x^2\cos(\ln(x)(1/x) \\ &= 2x\sin(\ln(x)) + x\cos(\ln(x)).\end{aligned}$$