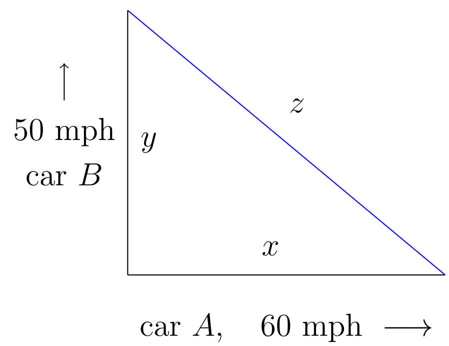
Math 111 lecture for Friday, Week 5

RELATED RATES

Example 1. Cars A and B travel at right angles, starting at the same point. Car A travels at 60 mph and car B travels at 50 mph. How fast is the distance between the cars increasing at 2 hours?

SOLUTION: The **first step** in a related rates problem is to draw a picture of the situation, labeling all relevant quantities. In our case, the picture could be as below:



The **second step** is to write equations relating the relevant variables and stating what we are given in terms of the variables. In our case, we use the Pythagorean theorem and express the speed of the cars:

$$x^2 + y^2 = z^2, \qquad \frac{dx}{dt} = 60, \qquad \frac{dy}{dt} = 50.$$

Then restate our problem in terms of our variables. In our case, the problem becomes:

$$\text{Find } \frac{dz}{dt} \text{ when } t = 2.$$

The **third step** is to use the chain rule to differentiate the equation relating the variables. In our case, we differentiate $x^2 + y^2 = z^2$ with respect to t:

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt},$$

or, dividing by 2,

$$x\frac{dx}{dt} + y\frac{dy}{dt} = \frac{dz}{dt}, \tag{1}$$

The **fourth step** (and final) is to substitute into this equation all the quantities we know and solve for the quantity we are trying to determine. In our case, when $t = 2$, we have $x = 120$ and $y = 100$. From the equation $x^2 + y^2 = z^2$, we find

$$z = \sqrt{120^2 + 100^2} = 20\sqrt{61}.$$

Substituting into equation (1), we get

$$120 \cdot 60 + 100 \cdot 50 = 20\sqrt{61}\,\frac{dz}{dt}.$$

Solving for dz/dt, we get the answer: the cars are moving apart at

$$\frac{dz}{dt} = \frac{1}{20\sqrt{61}}(7200 + 5000) = \frac{1}{10\sqrt{61}}(3600 + 2500) =\approx 78.1\,\text{mph}.$$

Question. As the time goes on, are the cars moving apart at the same rate as when they are after 2 hours? Faster? Slower?

SOLUTION: Let's figure out how quickly cars A and B are moving apart at an arbitrary time t. The set-up is the same as above. Move we want to find dz/dt at and arbitrary time t. Again, we use equation (1). After t hours, we have $x = 60t$ and $y = 50t$. Hence,

$$z = \sqrt{(60t)^2 + (50t)^2} = t\sqrt{60^2 + 50^2} = 10t\sqrt{61}.$$

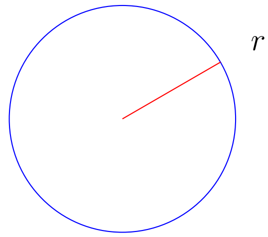
Plugging into equation (1):

$$(60t) \cdot 60 + (50t) \cdot 50 = 10t\sqrt{61}\,\frac{dz}{dt}.$$

Canceling the t on both sides of this equation, we see that the value for dz/dt, in fact, does not depend on t. So the cars are forever moving apart at the same rate of about 78.1 mph.

Example 2. Suppose that oil is spilled on water and spreads in a circular pattern so that the radius is increasing at 2 ft/sec. How fast is the area of the spill increasing when the radius is 60 ft?

SOLUTION: **First step**—draw the relevant picture an assign names to the relevant variables:



Second step—write equations relating the relevant variables and stating what we are given in terms of the variables. We are asked to relate the radius to the area. So let A denote the area of the circle. The only relevant equation relating the variables is $A = \pi r^2$. In terms of these variables, we are being asked to find dA/dt when $r = 60$, and we are given that $dr/dt = 2$.

Third step—differentiate the equation relating the variables. Differentiating $A = \pi r^2$ with respect to time gives

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt} = 2\pi r \cdot 2 = 4\pi r.$$

Fourth step—substitute into this equation all the quantities we know and solve for the quantity we are trying to determine. When $r = 60$, we have $dA/dt = 4\pi \cdot 60 = 240\pi$ ft/sec.

Example 3. The ideal gas law states that

$$PV = nRT$$

where P is the pressure of the gas, V is its volume, n is the amount (in moles), R is a constant, and T is the absolute temperature. Suppose we have gas in a cylinder with a piston on one end. By moving the piston in and out, we can adjust the volume of the cylinder. Suppose we do this but maintain a constant temperature using some kind of apparatus attached to the outside of the cylinder. The ideal gas law then says that

$$PV = \text{ constant.}$$

So as we decrease the volume by pressing in on the piston, the pressure will increase. Suppose we decrease the volume at a constant rate. Does the pressure increase at a constant rate?

SOLUTION: Suppose that $PV = c$ for some constant c. We are thinking of P and V as changing over time. So $P = P(t)$ and $V = V(t)$, i.e., both are functions of time. If the volume is decreasing at a constant rate, we can write

$$\frac{dV}{dt} = -k$$

for some $k > 0$. Take the derivative of $PV = c$ with respect to t using the product rule and the chain rule:

$$\begin{aligned} \frac{d}{dt}(PV) = \frac{d}{dt}(c) \quad &\Rightarrow \quad \frac{dP}{dt} \cdot V + P \cdot \frac{dV}{dt} = 0 \\ &\Rightarrow \quad \frac{dP}{dt} \cdot V - P \cdot k = 0. \end{aligned}$$

Solving for the rate of change of P over time:

$$\frac{dP}{dt} = \frac{P}{V} \cdot k.$$

Now solve for P in $PV = c$ and substitute into the above equation to get

$$\frac{dP}{dt} = \frac{c}{V^2} \cdot k.$$

So as the volume gets smaller, the pressure increases not at a constant rate but according to an inverse-square law.