Math 111 lecture for Wednesday, Week 4

Warm-up and review. Suppose the position of a particle along the y-axis is given by $f(x) = \sqrt{x}$.

1. What is the average speed of the particle between times $x = 1$ and $x = 4$?

   SOLUTION:

$$\text{average speed} = \frac{f(4) - f(1)}{4 - 1} = \frac{\sqrt{4} - \sqrt{1}}{3} = \frac{1}{3}.$$

2. What is the instantaneous speed of the particle at time $x = 1$?

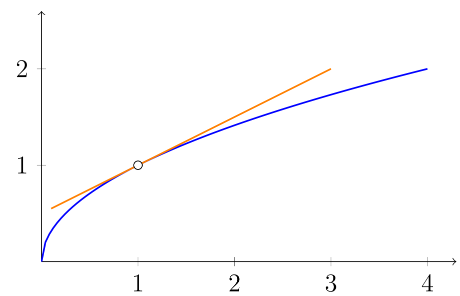
   SOLUTION:

$$\begin{aligned} f'(1) &= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h \to 0} \frac{\sqrt{1+h} - 1}{h} \\ &= \lim_{h \to 0} \frac{\sqrt{1+h} - 1}{h} \cdot \frac{\sqrt{1+h} + 1}{\sqrt{1+h} + 1} \\ &= \lim_{h \to 0} \frac{(1+h) - 1}{h(\sqrt{1+h} + 1)} \\ &= \lim_{h \to 0} \frac{h}{h(\sqrt{1+h} + 1)} \\ &= \lim_{h \to 0} \frac{1}{\sqrt{1+h} + 1} \\ &= \lim_{h \to 0} \frac{1}{2}. \end{aligned}$$

3. Find the equation for the tangent line to f at $x = 1$.

SOLUTION: We have just calculated the slop of that line: $f'(1) = 1/2$. Therefore, the line has equation

$$\frac{y - f(1)}{x - 1} = f'(1) = \frac{1}{2} \Rightarrow y = 1 + \frac{1}{2}(x - 1) \Rightarrow y = 1 + \frac{1}{2}x - \frac{1}{2}.$$



Graph of $f(x) = \sqrt{x}$ and its tangent line at $x = 1$.

4. What is the instantaneous speed at an arbitrary time x.

SOLUTION:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\
&= \lim_{h\to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} \\
&= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \\
&= \lim_{h\to 0} \frac{1}{2\sqrt{x}}.
\end{aligned}$$

5. What are $\lim_{x\to 0^+} f'(x)$ and $\lim_{x\to\infty} f'(x)$, and what does this mean geometrically?

   SOLUTION: We won't prove this, but since $f'(x) = 1/2\sqrt{x}$, it turns out that $\lim_{x\to 0^+} f'(x) = \infty$ and $\lim_{x\to\infty} f'(x) = 0$. Note from the graph of $f(x)$, drawn above, that the slope of the graph approaches ∞ as $x \to 0^+$ and approaches 0 as $x \to \infty$.

First properties of derivatives.

Recall our limit theorem which told us how to build up complicated limits from the limits of simple functions. There is a similar result for derivatives, but it has a very interesting twist when it comes to products of functions.

Theorem. Suppose f and g are differentiable functions at a point x.

1. The derivative of a constant function is 0: Let $c \in \mathbb{R}$, and let $h(x) = c$. or written in different notation, $(c)' = 0$. (Note that the graph of a constant function is a straight line with slope 0. So this makes sense.)

2. Let $k(x) = x$. Then $k'(x) = 1$, i.e., $(x)' = 1$. (Note that the graph of k is a line with slope 1.)

3. $(f(x)+g(x))' = f'(x)+g'(x)$: the derivative of a sum is the sum of the derivatives.

4. The *product rule or Leibniz rule.*
$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x).$$

5. The *quotient rule.*
$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}.$$

We'll prove parts of this theorem later. They follow straight from the definition of the derivative by taking limits. For now, let's play with the theorem to see what it tells us. First, some notation to save time. Instead of writing $(f(x)g(x))' = f'(x)g(x) + f(x)g(x)$, we'll often write $(fg)' = f'g + fg'$, dropping the x. Similarly, we'll write
$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}.$$

One consequence of the theorem is that when we compute derivatives we can "pull constants out" in the following sense: Let c be a constant. Then by the product rule:

$$(cf)' = (c)'f + cf' = 0 \cdot f + cf' = cf'.$$

For example,

$$(5x)' = 5(x)' = 5 \cdot 1 = 5.$$

Here is another example of a consequence of the theorem:

$$(f - g)' = f' + (-g)' = f' + (-1 \cdot g)'.$$

Continuing, using the fact that we can pull constants out:

$$= f' - 1 \cdot (g)' = f' - g'.$$

So $(f - g)' = f' - g'$.

Finally, note that the function $f(x) = x^n$ where n is any number $n = 1, 2, \ldots$ can be written as a product of functions: $x^n = x \cdot x \cdots x$. So we can apply the product rule to find its derivative. For instance,

$$(x^2)' = (x \cdot x)' = (x')x + x(x)' = 1 \cdot x + x \cdot 1 = 2x.$$

Knowing that $(x^2)' = 2x$, we find

$$(x^3)' = (x^2 \cdot x) = (x^2)'x + x^2 \cdot (x)' = 2x \cdot x + x^2 \cdot 1 = 3x^2.$$

Knowing that $(x^3)' = 3x^2$, we find

$$(x^4)' = (x^3 \cdot x) = (x^3)'x + x^3 \cdot (x)' = 3x^2 \cdot x + x^3 \cdot 1 = 4x^3.$$

Continuing in this way we get

$$(x^n) = nx^{n-1}$$

for $n = 1, 2, \ldots$

Combining that result with the derivative theorem, we can now compute the derivative of any polynomial (or quotient of polynomials). For example:

$$\begin{aligned}(3x^5 - 2x^2 - 7)' &= (3x^5)' + (-2x^2)' + (-7)' \\ &= 3(x^5)' - 2(x^2)' + 0 \\ &= 3(5x^4) - 2(2x) \\ &= 15x^4 - 4x.\end{aligned}$$