Math 111 lecture for Monday, Week 4

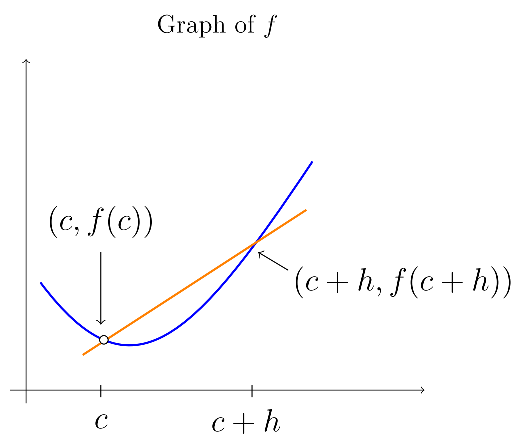
THE DERIVATIVE

Definition. The *derivative* of the function f at the point c is

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h},$$

provided the limit exists. If it does, then f is *differentiable at* c. If f is differentiable at all point where it is defined, then we say that f is *differentiable*. The derivative at $x = c$ is also known as the *slope of* f *at* c or the *instantaneous rate of change of* f *at* c.

The picture below illustrated the geometric meaning of the derivative:



The slope of the orange *secant* line is

$$\frac{f(c+h) - f(c)}{h}.$$

This is the *average rate of change* of f from $x = c$ to $x = c + h$. If f is thought of as giving the distance of a particle along the y-axis, then the quotient above is the *average speed* between times $x = c$ and $x = c + h$. To picture the derivative, imagine what happens to the above picture as h becomes smaller and smaller. The orange secant line will get less and less steep (for the function pictured). In the limit, we will get a line, called the *tangent to* f *at* c that touches the graph just at the point $(c, f(c))$. Its slope is the derivative.

Estimating $f'(x)$. Suppose we have the following table of values for a function f:

x	0.99	1.00	1.01	1.10
$f(x)$	1.8	2	2.03	2.1

To estimate $f'(1)$, we compute the average rate of change over an interval near $x = 1$, i.e., use the approximation

$$f'(1) \approx \frac{f(1+h) - f(h)}{h}$$

for h small. For example, with $h = 0.1$,

$$f'(1) \approx \frac{f(1+0.1) - f(1)}{0.1} = \frac{2.1 - 2}{0.1} = 1,$$

or better (probably), we could take $h = 0.01$ to get

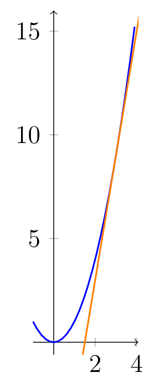
$$f'(1) \approx \frac{f(1+0.01) - f(1)}{0.01} = \frac{2.03 - 2}{0.01} = 3.$$

Calculating the derivative using the definition.

1. Find the derivative of $f(x) = x^2$ at $x = 3$.

   SOLUTION:

$$\begin{aligned} f'(3) &= \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} \\ &= \lim_{h\to 0} \frac{(3+h)^2 - 3^2}{h} \\ &= \lim_{h\to 0} \frac{(9+6h+h^2) - 9}{h} \\ &= \lim_{h\to 0} \frac{6h+h^2}{h} \\ &= \lim_{h\to 0}(6+h) \\ &= 6. \end{aligned}$$



Graph of $f(x) = x^2$ and its tangent line at $x = 3$.

2. Find the derivative of $f(x) = x^2$ at an arbitrary point x.

SOLUTION:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{(x^2 + 2hx + h^2) - x^2}{h} \\ &= \lim_{h\to 0} \frac{2hx + h^2}{h} \\ &= \lim_{h\to 0} (2x + h) \\ &= 2x. \end{aligned}$$

Thus, $f'(x) = 2x$. Notation: we may also write $(x^2)' = 2x$. Letting $x = 3$ gives our previous result: $f'(3) = 2 \cdot 3 = 6$. In general, $f'(x)$ gives the slope of f. The graph of f is a parabola, and it makes sense that its slope increases as x increases. Our formula shows that wherever $x < 0$, the slope is negative, and wherever $x > 0$, the slope is positive. At $x = 0$ the slope is $f'(0) = 0$, which also makes sense: the tangent line there is horizontal.

3. Find the derivative of $f(x) = 5x + 7$ at an arbitrary point x.

SOLUTION:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{(5(x+h)+7) - (5x+7)}{h} \\ &= \lim_{h \to 0} \frac{5h}{h} \\ &= \lim_{h \to 0} 5 \\ &= 5. \end{aligned}$$

Thus, $f'(x) = 5$ at every point x. This says the slope of the function is 5 as every point, which makes sense since the graph of f is a line with slope 5.

4. Find the derivative of $f(x) = x^n$ at an arbitrary point x.

SOLUTION:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{(x+h)^n - x^n}{h} \\ &= \lim_{h \to 0} \frac{(x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n) - x^n}{h} \\ &= \lim_{h \to 0} \frac{nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n}{h} \\ &= \lim_{h \to 0} (nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1}) \\ &= nx^{n-1}. \end{aligned}$$

For example the derivative of the function $f(x) = x^5$ at an arbitrary point x is $5x^4$. For instance, the slope of $f(x) = x^5$ at the point 2 is $5 \cdot 2^4 = 80$. The case $n = 2$ recaptures our previous result for $f(x) = x^2$, i.e., it's derivative is $2x$.

Equation of the tangent line. The *tangent line* for $f(x)$ at a point $x = c$ is the line with slope $f'(c)$ and passing through the point $(c, f(c))$. It has the equation

$$\frac{y - f(c)}{x - c} = f'(c),$$

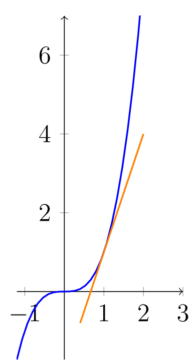
or, solving for y:

$$y = f(c) + f'(c)(x - c).$$

Example. To find the tangent line to $f(x) = x^3$ at the point $x = 1$, we first calculate the derivative of $f(x) = x^3$ at $x = 1$ using the formula we computed above: $f'(x) = 3x^2$, hence, the slope is $f'(1) = 3$. The tangent line has equation

$$y = f(1) + f'(2)(x - 1) = 1 + 3(x - 1) = 3x - 2.$$

In sum, the tangent line at $x = 1$ has equation $y = 3x - 2$.



Graph of $f(x) = x^3$ and its tangent line at $x = 1$.