Math 111 lecture for Friday, Week 4

Our goal today is to prove the "derivative theorem" presented last time. We'll need the definition of the derivative:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists, and we'll need our earlier limit theorem:

Limit Theorem. Suppose $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$. Then

1. $\lim_{x\to c}(f(x) + g(x)) = L + M$,
2. $\lim_{x\to c} f(x)g(x) = LM$,
3. if $M \neq 0$, then
$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M}.$$

Derivative theorem. Suppose f and g are differentiable functions at a point x. Then

1. $(f(x) + g(x))' = f'(x) + g'(x)$,
2. *product rule or Leibniz rule*:
$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x).$$
3. *quotient rule*:
$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}.$$

Proof. Part 1:

$$\begin{aligned}
(f(x)+g(x))' &= \lim_{h\to 0}\frac{(f(x+h)+g(x+h))-(f(x)+g(x))}{h}\\
&= \lim_{h\to 0}\frac{(f(x+h)-f(x))+(g(x+h)-g(x))}{h}\\
&= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}\right)\\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\
&= f'(x)+g'(x).
\end{aligned}$$

Notice how we used our earlier limit theorem to say that the limit of a sum is the sum of the limits.

Part 2. This one's a bit trickier—it involves subtracting and adding $f(x)g(x+h)$:

$$\begin{aligned}
(f(x)g(x))' &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}\\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x+h)+f(x)g(x+h)-f(x)g(x)}{h}\\
&= \lim_{h\to 0}\left(\frac{(f(x+h)-f(x))g(x+h)}{h}+\frac{f(x)(g(x+h)-g(x))}{h}\right)\\
&= \lim_{h\to 0}\frac{(f(x+h)-f(x))g(x+h)}{h}+\lim_{h\to 0}\frac{f(x)(g(x+h)-g(x))}{h}\\
&= \lim_{h\to 0}\frac{(f(x+h)-f(x))}{h}\lim_{h\to 0}g(x+h)+\lim_{h\to 0}f(x)\lim_{h\to 0}\frac{(g(x+h)-g(x))}{h}.
\end{aligned}$$

For the last two steps in the above calculation, we used parts 1 and 2 of the limit theorem. Continuing, now use the definition of the derivative:

$$(f(x)g(x))' = f'(x)\lim_{h\to 0}g(x+h)+\left(\lim_{h\to 0}f(x)\right)g'(x).$$

We have $\lim_{h\to 0} f(x) = f(x)$ since $f(x)$ can be thought of as a constant function of h. Therefore,

$$(f(x)g(x))' = f'(x)\left(\lim_{h\to 0} g(x+h)\right) + f(x)g'(x).$$

Finally, we use a fact the we may or may not prove later: differentiable functions are continuous, i.e., we can evaluate their limits by just plugging in the limit point. In particular, g is continous. Then $g(x+h)$ is a composition of continuous functions of h: $g(x+h) = g(k(h))$ where $k(h) = x+h$. Therefore, $\lim_{h\to 0} g(x+h) = g(x+0) = g(x)$. So we finally get

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x).$$

For Part 3, we first leave the following as an exercise for the reader:

$$\left(\frac{1}{g(x)}\right)' = -\frac{g'(x)}{g^2(x)}$$

where $g^2(x) = g(x)\cdot g(x)$. We combine this with the product rule of Part 1 to get

$$\begin{aligned}
\left(\frac{f(x)}{g(x)}\right)' &= \left(f(x)\cdot\frac{1}{g(x)}\right)' \\
&= f'(x)\cdot\frac{1}{g(x)} + f(x)\left(\frac{1}{g(x)}\right)' \\
&= \frac{f'(x)}{g(x)} + f(x)\left(-\frac{g'(x)}{g^2(x)}\right) \\
&= \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)},
\end{aligned}$$

where in the last step, we've just found a common denominator. □

Last time, we saw that this theorem makes calculating derivatives a lot easier than having to go back to the definition of the derivative every time. We saw that knowing $(c)' = 0$ for a constant $c \in \mathbb{R}$, and $(x)' = 1$, we can use the theorem to compute the derivative of any rational function. For instance, repeated use of the product rule allowed us the compute $(x^n)' = nx^{n-1}$

for $n = 1, 2, \ldots$ Using that fact and the derivative theorem, we can evaluate derivatives of all polynomials. For example,

$$\begin{aligned}\left(3x^4 + x^2 - 4x + 2\right)' &= (3x^4)' + (x^2)' + (-4x)' + (2)' \qquad \text{(Part 1)}\\ &= 3(x^4)' + (x^2)' - 4(x)' + 0 \quad \text{(pulling out constants—see the last lecture)}\\ &= 3(4x^2) + (2x) - 4(1)\\ &= 12x^2 + 2x - 4.\end{aligned}$$

With practice, you can perform all of these steps at once:

$$(6x^5 - 4x^3 + 12x^2 - 7x + 2)' = 30x^4 - 12x^2 + 24x - 7.$$

Now we apply the quotient rule to compute $(x^n)'$ for $n < 0$. To start,

$$(x^{-1})' = \left(\frac{1}{x}\right)' = -\frac{(x)'}{x^2} = -\frac{1}{x^2}.$$

Using this, we get

$$(x^{-2})' = \left(\frac{1}{x^2}\right)' = -\frac{(x^2)'}{(x^2)^2} = -\frac{2x}{x^4} = -\frac{2}{x^3}.$$

Continuing in this way, we get

$$\left(\frac{1}{x^n}\right)' = -\frac{n}{x^{n+1}}.$$

for $n = 1, 2, 3, \ldots$

Here is an example of a computation of the derivative of a typical rational function using the quotient rule:

$$\begin{aligned}\left(\frac{x^2}{x^4 + 3x + 2}\right)' &= \frac{(x^2)'(x^4 + 3x + 2) - x^2(x^4 + 3x + 2)'}{(x^4 + 3x + 2)^2}\\ &= \frac{2x(x^4 + 3x + 2) - x^2(4x^3 + 3)}{(x^4 + 3x + 2)^2}\\ &= \frac{2x^5 + 6x^2 + 4x - 4x^5 - 3x^2}{(x^4 + 3x + 2)^2}\\ &= \frac{-2x^5 + 3x^2 + 4x}{(x^4 + 3x + 2)^2}.\end{aligned}$$