Math 111 lecture for Wednesday, Week 3

Last time, we used the rationalization trick to show that

$$\lim_{x\to 6}\frac{\sqrt{x+3}-3}{x-6}=\frac{1}{6}.$$

At one point in the calculation we needed to use the fact that

$$\lim_{x\to 6}(\sqrt{x+3}+3)=6.$$

Our immediate goal is prove that fact now, using it as an excuse to introduce two important ideas: (I) continuity and (II) limits of compositions of functions.

I. Continuity. The first instinct in evaluating $\lim_{x\to c} f(x)$ is to just stick in c for x. In other words, there is a natural tendency to say $\lim_{x\to c} f(x) = f(c)$. For most of the functions you've dealt with in the past, this way of thinking is valid, and if these were the only kind of functions, there would be no need for the concept of a limit. These best-behaved functions are called *continuous* functions.

Definition. The function f is *continuous at a point* $c\in\mathbb{R}$ if

$$\lim_{x\to c} f(x) = f(c).$$

If f is continuous at every point, we simply say f is a *continuous* function.

Since $\lim_{x\to c} x = c$, another way to say f is continuous is to say $\lim_{x\to c} f(x) = f(\lim_{x\to c})$ for all c. In other words, you can bring the limit sign inside. We say "f commutes with limits".

Examples of continuous functions include any polynomial or quotient of polynomials (if you don't divide by zero). This is easily proved with our Limit Theorem. The following functions are also continuous, and from now on, we will accept that fact without proof:

$$\sqrt{x},\ e^x,\ \cos(x),\ \sin(x).$$

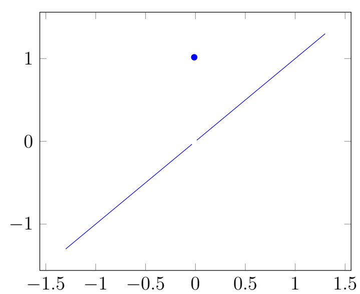
The following function is not continuous:

$$f(x)=\begin{cases} x & \text{if } x\neq 0\\ 1 & \text{if } x=0.\end{cases}$$

To check it's not continuous, first note that if $x \neq 0$, we have $f(x) = x$, thus, taking the limit as $x \to 0$, we have

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} x = 0 \neq f(0) = 1.$$

We are allowed the first equal sign in the above calculation because we never consider the case $x = 0$ when evaluating the limit as $x \to 0$. (Recall that in the definition of the limit, we consider x such that $0 < |x - c| < \delta$. So we don't care what happens when $x = c$.) Since $\lim_{x \to 0} f(x) \neq f(0)$, the function is not continuous at 0. We can see the discontinuity of f at 0 in the graph of f, below:



Graph of the function $f(x)$ defined above.

II. Composition of functions. Recall that if $f(x)$ and $g(x)$ are functions, then the *composition* of f and g is the function:

$$(f \circ g)(x) := f(g(x)).$$

(Note: the symbol ":=" means "the thing on the left is defined by the thing on the right".) So to compose f and g, you first evalute g, then stick the result into f.

Examples. (i) If $f(x) = x^3$ and $g(x) = x + 2$, then

$$(f \circ g)(x) = f(g(x)) = f(x + 2) = (x + 2)^3.$$

(ii) If $f(x) = \sqrt{x}$ and $g(x) = x + 3$, then

$$(f \circ g)(x) = f(g(x)) = f(x + 3) = \sqrt{x + 3},$$

and

$$(g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = \sqrt{x} + 3.$$

Composition theorem. If f and g are continuous functions, then so is $f \circ g$:

$$\lim_{x \to c} (f \circ g)(x) = f(g(c))$$

for all c.

Proof. Math 112. $\square$

We say "a composition of continuous functions is continuous".

Example. We will show that $\lim_{x \to 6} \sqrt{x+3} = 3$ using I and II, above. Let $f(x) = \sqrt{x}$ and $g(x) = x + 3$. The function we are interested in is $(f \circ g)(x) = f(x+3) = \sqrt{x+3}$. Then since f and g are continuous functions, so is the composition. It follows that

$$\lim_{x \to 6} \sqrt{x+3} = \lim_{x \to 6} (f \circ g)(x) = (f \circ g)(6) = f(g(6)) = f(9) = 3.$$