Math 111 lecture for Monday, Week 3

Last time, we introduced the Limit Theorem, which allows us to give "high level" limit proofs, i.e., limit proofs that don't go all the way back to the definition, using εs and δs. The strategy for evaluating limits using the limit theorem is:

1. First compute the limits of some simple functions by hand. In our case, we showed how to evaluate the limit of a constant function and of the function $f(x) = x$.

2. Next, the limit theorem shows us how to compute limits for any function that we can build out of these simple functions using addition, multiplication, and division. Given constant function and $f(x) = x$ as our building blocks, for example, we can create all rational functions, i.e., all functions of the form $g(x)/h(x)$ where g and h are polynomials.

We will now introduce two important techniques for evaluating limits: the cancellation and the rationalization tricks.

Cancellation trick. Last time, we saw a typical example of using the limit theorem to evaluate the limit of a rational function (quotient of polynomials). Here, we'll give a similar example but where the limit theorem seems to break down.

$$\begin{aligned}
\lim_{x\to 2}\frac{x^2+x-6}{x-2} &= \frac{\lim_{x\to 2}(x^2+x-6)}{\lim_{x\to 2}(x-2)}\\
&= \frac{\lim_{x\to 2}x^2+\lim_{x\to 2}x+\lim_{x\to 2}(-6)}{\lim_{x\to 2}x+\lim_{x\to 2}(-2)}\\
&= \frac{\lim_{x\to 2}x\lim_{x\to 2}x+\lim_{x\to 2}x+\lim_{x\to 2}(-6)}{\lim_{x\to 2}x+\lim_{x\to 2}(-2)}\\
&= \frac{2\cdot 2+2-6}{2-2}\\
&= \frac{0}{0}.
\end{aligned}$$

What's the problem? The answer is that part 3 of the limit theorem says that

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$$

provided $\lim_{x \to c} g(x) \neq 0$. So the very first step of the argument above is not allowed since $\lim_{x \to 2}(x - 2) = 0$. Worse, it turns out that it is limits like this one that are most important in the study of calculus. Recall that in computing instantaneous speed for a distance function $f(t)$ at time $t = a$, we first compute the average speed over a time interval h:

$$\text{average\_speed}(h) = \frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h},$$

then we take the limit as $h \to 0$. However, trying to apply the limit theorem, a similar problem arises since $\lim_{h \to 0} h = 0$ in the denominator. So we can't directly use the limit theorem.

We now correctly calculate the limit in the example to show a typical way forward in this situation:

$$\begin{aligned}
\lim_{x \to 2} \frac{x^2 + x - 6}{x - 2} &= \lim_{x \to 2} \frac{(x-2)(x+3)}{x - 2} \\
&= \lim_{x \to 2} (x + 3) \\
&= \lim_{x \to 2} x + \lim_{x \to 2} 3 \\
&= 2 + 3 \\
&= 5.
\end{aligned}$$

In the second step, we replace the function

$$\frac{(x-2)(x+3)}{x-2}$$

with the function

$$x + 3.$$

The reason this is OK is that these two functions are equal for all x except $x = 2$. When $x = 2$, the first function is undefined while the second function is $2 + 3 = 5$. *However*, when calculating the limit, recall that we only worry about x such that $0 < |x - c| < \delta$. Since $0 < |x - c|$, this means we never consider the case where $x = c$, or in our case, $x = 2$. So for the purpose of our limit, we are allowed to substitute $x+3$ for $(x-2)(x-3)/(x-3)$.

Rationalization trick. The cancellation trick, just illustrated above, comes up fairly often in calculus. Here is another, somewhat less common, technique. It relies on the fact that $(a-b)(a+b) = a^2 - b^2$ and that multiplying the top and the bottom of a fraction by the same thing does not change the value of the fraction (since it amounts to multiplying by 1). We will need to use the fact that $\lim_{x\to 6}\sqrt{x+3}+3 = 6$, which we'll just assume for now (it should seem reasonable) and prove later.

Consider the limit

$$\lim_{x\to 6}\frac{\sqrt{x+3}-3}{x-6}.$$

The first thing you should consider is a straightforward application of our limit theorem. That amounts plugging in 6 for x. Unfortunately, $\lim_{x\to 6}(x-6) = 0$ in the denominator. However, there is hope for cancellation since $\lim_{x\to 6}(\sqrt{x+3}-3) = 0$, too. Here is how the computation goes:

$$\begin{aligned}
\lim_{x\to 6}\frac{\sqrt{x+3}-3}{x-6} &= \lim_{x\to 6}\frac{\sqrt{x+3}-3}{x-6}\cdot\frac{\sqrt{x+3}+3}{\sqrt{x+3}+3} \\
&= \lim_{x\to 6}\frac{(\sqrt{x+3}-3)(\sqrt{x+3}+3)}{(x-6)(\sqrt{x+3}+3)} \\
&= \lim_{x\to 6}\frac{(x+3)-9}{(x-6)(\sqrt{x+3}+3)} \\
&= \lim_{x\to 6}\frac{x-6}{(x-6)(\sqrt{x+3}+3)} \\
&= \lim_{x\to 6}\frac{1}{\sqrt{x+3}+3} \\
&= \frac{1}{6}.
\end{aligned}$$

We now give a proof of part 1 of the Limit Theorem: if $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist, then

$$\lim_{x\to c}(f(x)+g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x).$$

We gave the motivation for the proof in the last lecture (and you might want to review that motivation before continuing). The proof will include the important "$\varepsilon/2$-trick" and depends on the following workhorse of analysis:

Triangle inequality. Let x and y be real numbers. Then

$$|x+y| \leq |x| + |y|.$$

PROOF. Math 112. $\square$.

For example,

$$3 = |-3+6| \leq |-3| + |6| = 3+6 = 9.$$

Proof of part 1 of the limit theorem. Suppose $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$. We must show that

$$\lim_{x\to c}(f(x)+g(x)) = L+M.$$

Let $\varepsilon > 0$. We need to show that there is a $\delta > 0$ such that $0 < |x-c| < \delta$ implies

$$|(f(x)+g(x)) - (L+M)| < \varepsilon.$$

Since $\lim_{x\to c} f(x) = L$, we know that given any $\varepsilon' > 0$, there is a $\delta' > 0$ such that $0 < |x-c| < \delta'$ implies $|f(x)-L| < \varepsilon'$. In particular, there is such a δ' in the case where $\varepsilon' = \varepsilon/2$ (where ε is the number we need to beat, fixed above). To summarize: there exists $\delta' > 0$ such that $0 < |x-c| < \delta'$ implies

$$|f(x)-L| < \frac{\varepsilon}{2}.$$

By the same argument, since $\lim_{x\to c} g(x) = M$, there exists a $\delta'' > 0$ such that $0 < |x-c| < \delta''$ implies

$$|g(x)-M| < \frac{\varepsilon}{2}.$$

Recall that once we have one δ that works in the definition of the limit, then we can replace it by any smaller (positive) δ. So define $\delta = \min\{\delta', \delta''\}$, the minimum of δ' and δ''. It follows that if $0 < |x - c| < \delta$ we have both inequalities

$$|f(x) - L| < \frac{\varepsilon}{2} \quad \text{and} \quad |g(x) - M| < \frac{\varepsilon}{2},$$

simultaneously. Using the triangle inequality, we then see that $0 < |x-c| < \delta$ implies

$$\begin{aligned} |(f(x) + g(x)) - (L - M)| &= |(f(x) - L) + (g(x) - M)| \\ &\leq |f(x) - L| + |g(x) - M| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon, \end{aligned}$$

as required. $\square$