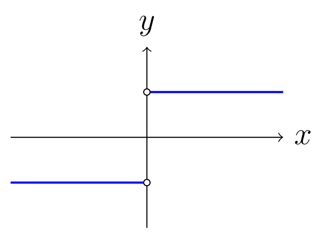
Math 111 lecture for Friday, Week 3

Today, we'll talk about two things: variations on the definition of the limit, and the intermediate value theorem (IVT).

I. Variations on the definition of the limit.

Right- and left-hand limits. We'll start with an example. Consider the function

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0. \end{cases}$$



Graph of $f(x)$.

As you might guess from the picture, $\lim_{x\to 0} f(x)$ does not exists (for instance, you can't beat $\varepsilon = 2$ with any δ). However, only think of the right-hand side of this graph, the limit looks like it should be 1, and if you only think of the left-hand side, the limit looks like it should be -1. The following definitions make this intuition precise.

Definition. *Right-hand limit:* $\lim_{x\to c^+} f(x) = L$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that if x satisfies $0 < |x - c| < \delta$ and $c < x$, then $|f(x) - L| < \varepsilon$.

Left-hand limit $\lim_{x\to c^-} f(x) = L$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that if x satisfies $0 < |x - c| < \delta$ and $x < c$, then $|f(x) - L| < \varepsilon$.

Remark. Not that these definitions look very similar to the ordinary definition of the limit. The only difference is two conditions are placed on x: first, $0 < |x - c| < \delta$, just as before, and second x is restricted to lie to either the right of c (i.e., $c < x$) or to the left of c (i.e., $x < c$). There is a nice way we can combine the two conditions on x into one expression. For the case of right-hand limits, note that the condition $c < x$ is equivalent to $0 < x - c$, which is equivalent to $|x - c| = x - c$. So

$$c < x \quad \text{and} \quad 0 < |x - c| < \delta$$

is equivalent to

$$0 < x - c < \delta.$$

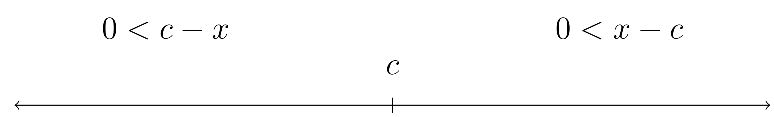
Similarly, $x < c$ is equivalent to $0 < c - x$, which is equivalent to $|x - c| = c - x$. So

$$c < x \quad \text{and} \quad 0 < |x - c| < \delta$$

is equivalent to

$$0 < x - c < \delta.$$

Here is a picture of the situation on the number line:



In the ordinary definition of the limit, the x-values can be to the right or to the left of c, but for one-sided limits, they are restricted to one side of c.

The following theorem is useful in proving a function does not have a limit.

Theorem. Let f be a function defined in an interval about a point c. Then $\lim_{x \to c} f(x)$ exists if and only if $\lim_{x \to c^+} f(x)$ and $\lim_{x \to c^-} f(x)$ exist and are equal. In that case,

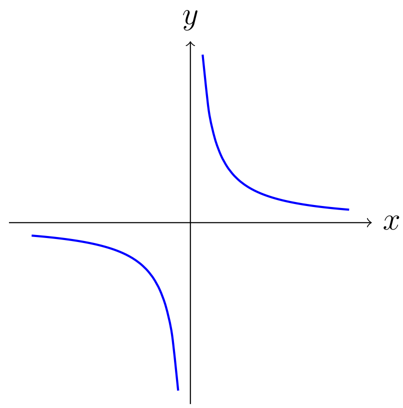
$$\lim_{x \to c} f(x) = \lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x).$$

Example. For the function f given at the beginning of this lecture, we have

$$\lim_{x \to 0^+} f(x) = 1 \quad \text{and} \quad \lim_{x \to 0^-} f(x) = -1.$$

Since the right- and left-hand limits are not equal, our theorem says that $\lim_{x \to 0} f(x)$ does not exist.

Limits at ∞*.* Below, we graph the function $g(x) = 1/x$:



Graph of $g(x) = 1/x$.

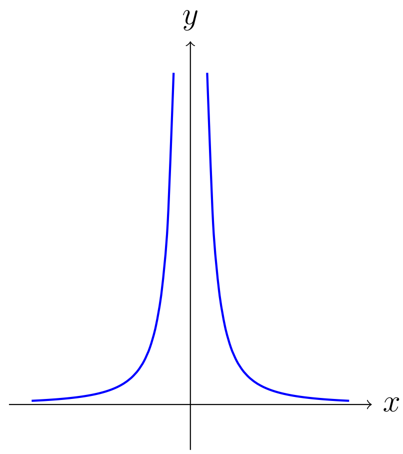
As x gets very large, in the positive or negative direction, the function $g(x) = 1/x$ gets very close to 0. So it is tempting to take the limit as x goes to ∞ or to $-\infty$. The trouble with trying to use our definition of the limit to make sense of the statement $\lim_{x\to\infty} = 0$ is that the condition $0 < |x-c| < \delta$ would become $0 < |x - \infty| < \delta$, which does not make sense. What would $x - \infty$ mean? So to make our intuition precise requires a different definition, which we give below:

Definition. We say $\lim_{x\to\infty} f(x) = L$ if for all $\varepsilon > 0$, there exists N such that if $x > N$, then $|f(x) - L| < \varepsilon$. We say $\lim_{x\to-\infty} f(x) = L$ if for all $\varepsilon > 0$, there exists N such that if $x < N$, then $|f(x) - L| < \varepsilon$.

Example. With these definitions, we have

$$\lim_{x\to\infty} \frac{1}{x} = 0 \quad \text{and} \quad \lim_{x\to-\infty} \frac{1}{x} = 0.$$

Infinite limits. Now consider the function $h(x) = 1/x^2$, whose graph appears below:



Graph of $g(x) = 1/x$.

What happens as $x \to 0$? This situation is somewhat similar to the previous one. Here is the definition we need:

Definition. We say $\lim_{x \to c} f(x) = \infty$ if for all N, there exists $\delta > 0$ such that $0 < |x - c| < \infty$ implies $f(x) > N$. We say $\lim_{x \to c} f(x) = -\infty$ if for all N, there exists $\delta > 0$ such that $0 < |x - c| < \infty$ implies $f(x) < N$.

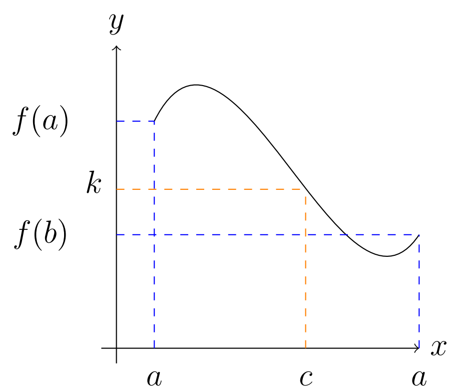
Remark. The condition $f(x) > N$ should be thought of as saying "$f(x)$ is really large. Similarly, $f(x) < N$ should be thought of a saying that "$f(x)$ is really negative.

Example. With these definitions, we have

$$\lim_{x \to 0} \frac{1}{x^2} = 0.$$

Intermediate Value Theorem (IVT). If f is a continuous function on a closed interval $[a, b]$, and k is a number between $f(a)$ and $f(b)$, then there exists $c \in [a, b]$ such that $f(c) = k$.

Proof. Math 112. $\square$



Graph of $f(x)$.

Corollary. Suppose f is continuous on $[a, b]$. If $f(a)$ and $f(b)$ have opposite signs, then there exists $c \in [a, b]$ such that $f(c) = k$.

Proof. If $f(a)$ and $f(b)$ have opposite signs, then $k = 0$ is between $f(a)$ and $f(b)$. Apply the IVT. $\square$

Example. Consider the polynomial $f(x) = x^5 + x + 1$. Then f is continuous since it's a polynomial. Since $f(-1) = -1$ and $f(0) = 1$, by the IVT, we know there is a $c \in [-1, 0]$ such that $f(c) = 0$. To find a more precise locate for a point where f is 0, compute $f(-0.5)$. We find $f(-0.5) = 0.46875$, which is positive. We know that $f(-1)$ is negative. Thus, by the IVT, there is a $c \in [-1, -0.5]$ such that $f(c) = 0$. Now evaluate f at the midpoint of $[-1, -0.5]$ and check out its sign. This helps narrow the locate of a zero of f even further. You can repeat this process, dividing an interval in half at each step to quickly approximate a zero of the function f.