Math 111 lecture for Wednesday, Week 2

Recall the definition of the limit of a function:

Definition. Let f be a function defined in an open interval containing a point c, except f might not be defined at the point c, itself. Let L be a real number. The *limit of $f(x)$ as x approaches c* is L, denoted $\lim_{x\to c} f(x) = L$, if for all $\varepsilon > 0$, there exists $\delta > 0$ such that if x satisfies

$$0 < |x - c| < \delta,$$

then

$$|f(x) - L| < \varepsilon.$$

Today, we will show how to use this definition.

Examples of limits.

Claim. $\lim_{x\to 7} 5x - 4 = 31$.

Proof. Given $\varepsilon > 0$, let $\delta = \varepsilon/5$. Suppose that $0 < |x - 7| < \delta = \varepsilon/5$. Then

$$|(5x - 4) - 31| = |5x - 35| = 5|x - 7| < 5 \cdot \frac{\varepsilon}{5} = \varepsilon,$$

as required. □

Claim. $\lim_{x\to 2} -3x - 1 = -7$.

Proof. Given $\varepsilon > 0$, let $\delta = \varepsilon/3$. Suppose that $0 < |x - 2| < \delta = \varepsilon/3$. Then

$$|(-3x - 1) - (-7)| = | - 3x + 6| = | - 3(x - 2)| = 3|x - 2| < 3 \cdot \frac{\varepsilon}{3} = \varepsilon,$$

as required. □

Suppose we have found one $\delta > 0$ such that $0 < |x - c| < \delta$ implies $|f(x) - L| < \varepsilon$. Then take $\delta' > 0$ such that $\delta' < \delta$. It follows that if x satisfies $0 < |x - c| < \delta'$, then $0 < |x - c| < \delta$, too, and hence, $|f(c) - L| < \varepsilon$. The point here is that **once you've found a suitable δ, you can always make δ smaller**. We use that fact in the following proof.

Claim. $\lim_{x\to 1} 6 - 1/x = 5$.

Proof. Given $\varepsilon > 0$, let $\delta = 0.5$ and suppose that $0 < |x - 1| < \delta = 0.5$. We have

$$|(6 - 1/x) - 5)| = |1 - 1/x| = \left|\frac{x-1}{x}\right| = |x - 1| \cdot \frac{1}{|x|}.$$

Since $|x - 1| < \delta = 0.5$, it follows that $0.5 < x < 1.5$. In particular, since $0.5 < x$, we have $1/|x| > 1/0.5 = 2$. Therefore,

$$|(6 - 1/x) - 5)| = |x - 1| \cdot \frac{1}{|x|} < 2|x - 1|.$$

Now replace δ by the minimum of $\varepsilon/2$ and 0.5, whichever is smallest. Suppose that $0 < |x - 1| < \delta$. Then, since $\delta \leq 0.5$, we still have that $1/|x| < 2$, and thus $|(6 - 1/x) - 5| < 2|x - 1|$. In addition, since $\delta \leq \varepsilon/2$, we have

$$|(6 - 1/x) - 5| < 2|x - 1| < 2 \cdot \frac{\varepsilon}{2} = \varepsilon,$$

as required. □

For instance, suppose we want to make the function $f(x) = 6 - 1/x$ within a distance of 0.1 of 5 by making x close to 1. Using the above proof with $\varepsilon = 0.1$, we see that we can take any $\delta > 0$ satisying $\delta \leq 0.5$ and $\delta \leq \varepsilon/2 = 0.1/2 = 0.05$. Thus, for any x satisfying $0 < |x - 1| < 0.05$, we have $|(6 - 1/x) - 5| < 0.1$.

The proof in next example is similar to that just seen. Note the simplicity of the function $f(x) = x^2$ and the seeming obviousness of the claim compared to difficulty of the proof!

Claim. $\lim_{x\to 5} x^2 = 25$.

Proof. Given $\varepsilon > 0$, let $\delta = \min\{1, \varepsilon/11\}$, i.e., δ is the minumum of 1 and $\varepsilon/11$. So $\delta \leq 1$ and $\delta \leq \varepsilon/11$ (with equality holding in at least one of these). Suppose that x satisfies $0 < |x - 5| < \delta$. Since $\delta \leq 1$, it follows $4 < x < 6$, and hence $9 < x + 5 < 11$. In particular, $|x + 5| < 11$. Therefore,

$$|x^2 - 25| = |(x + 5)(x - 5)| = |x + 5||x - 5| < 11|x - 5|.$$

Now, since $\delta \le \varepsilon/11$ and $|x-5| < \delta$, it follows that

$$|x^2 - 25| < 11|x-5| < 11 \cdot \frac{\varepsilon}{11} = \varepsilon,$$

as required. $\square$

So even with very simple functions like $f(x) = x^2$, these limit proofs are difficult. What is one to do? The answer is the following very general limit theorem:

Limit Theorem. Suppose that $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist. Then

1. $\lim_{x\to c}(f(x)+g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$.
2. $\lim_{x\to c} f(x)g(x) = \lim_{x\to c} f(x) \lim_{x\to c} g(x)$.
3. If $\lim_{x\to c} g(x) \neq 0$, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}.$$

In words, this theorem says that if we know the limits of f and g, then we can easily find the limit of any function we can construct from f and g using the operations of addition, multiplication, and division. For instance, combined with the following proposition, we can easily find the limit of any quotient of polynomials. We'll see that next time, but for a preview let's revisit a result we proved earlier:

Claim. $\lim_{x\to 5} x^2 = 25$.

Proof. It's easy to show $\lim_{x\to 5} x = 5$ (and we'll do this next time.) Then, using part 2 of the limit theorem with $f(x) = g(x) = x$, we get

$$\lim_{x\to 5} x^2 = \lim_{x\to 5}(x \cdot x) = \left(\lim_{x\to 5} x\right)\left(\lim_{x\to 5} x\right) = 5 \cdot 5 = 25.$$

$\square$

Notice what quick and clean work the limit theorem makes of this problem compared to our previous solution.