Math 111 lecture for Friday, Week 2

Last time we saw that even proving something intuitively obvious such as $\lim_{x\to 5} x^2 = 25$ using an ε- δ argument can be difficult. Imagine trying to prove that $\lim_{x\to 1} x^5 - x^3 + 2x^2 + 4 = 6$, also intuitively obvious, using an ε-δ argument. It might seem impossible. Today, we'll learn how to handle limits for all polynomials (and quotients of polynomials).

The strategy hinges on the following very general limit theorem:

Limit Theorem. Suppose that $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist. Then

1. $\lim_{x\to c}(f(x) + g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$.
2. $\lim_{x\to c} f(x)g(x) = \lim_{x\to c} f(x)\ \lim_{x\to c} g(x)$.
3. If $\lim_{x\to c} g(x) \neq 0$, then
$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}.$$

In words, this theorem says that if we know the limits of f and g, then we can easily find the limit of any function we can construct from f and g using the operations of addition, multiplication, and division. For instance, combined with the following proposition, we can easily find the limit of any quotient of polynomials.

Proposition. Let c and k be real numbers. Then

1. $\lim_{x\to c} k = k$.
2. $\lim_{x\to c} x = c$.

Proof. For part 1, let $f(x) = k$, the constant function. We are trying to show $\lim_{x\to c} f(x) = k$. Given $\varepsilon > 0$, let $\delta = 1$ (in fact, this argument will work for any choice of positive δ). Then $0 < |x - c| < \delta$ implies

$$|f(x) - k| = |k - k| = 0 < \varepsilon.$$

(This proof my be a little confusing because the condition we need to be satisfied, $|f(x) - k|$ is trivially satisfied since f is such a trivial function.)

For part 2, the function in question is $g(x) = x$. Given $\varepsilon > 0$, let $\delta = \varepsilon$ and suppose $0 < |x - c| < \delta = \varepsilon$. Then

$$|g(x) - c| = |x - c| < \delta = \varepsilon,$$

as required. (Again, pretty trivial since g is so simple.) □

To see how useful the above results are, let's go back to the function $f(x) = x^2$.

Example 1. $\lim_{x \to 5} x^2 = 25$.

Proof. By the above Proposition, we know $\lim_{x \to 5} x = 5$. Using part 2 of the limit theorem with $f(x) = g(x) = x$, we get

$$\lim_{x \to 5} x^2 = \lim_{x \to 5} (x \cdot x) = \left(\lim_{x \to 5} x \right) \left(\lim_{x \to 5} x \right) = 5 \cdot 5 = 25.$$

□

Much easier! Here is a useful result:

Claim ⋆. If $\lim_{x \to c} f(x)$ exists and $k \in \mathbb{R}$, then

$$\lim_{x \to c} k f(x) = k \lim_{x \to c} f(x).$$

Proof. Combine part 2 of the limit theorem with the proposition:

$$\lim_{x \to c} k f(x) = \lim_{x \to c} k \ \lim_{x \to c} f(x) = k \ \lim_{x \to c} f(x).$$

□

Example 2. $\lim_{x \to 1} \frac{3x^2 - 5}{x^3 - 2x + 3} = -1$.

Proof. Apply the limit theorem

$$\lim_{x\to 1}\frac{3x^2-5}{x^3-2x+3} = \frac{\lim_{x\to1}(3x^2-5)}{\lim_{x\to1}(x^3-2x+3)} \qquad \text{(part 3)}$$

$$= \frac{\lim_{x\to1} 3x^2 + \lim_{x\to1}(-5)}{\lim_{x\to1} x^3 + \lim_{x\to1} -2x + \lim_{x\to1} 3)} \qquad \text{(part 1)}$$

$$= \frac{3(\lim_{x\to1} x^2) - 5}{\lim_{x\to1} x^3 - 2\lim_{x\to1} x + \lim_{x\to1} 3} \qquad (\star \text{ and Prop.})$$

$$= \frac{3(\lim_{x\to1} x \lim_{x\to1} x) - 5}{\lim_{x\to1} x \lim_{x\to1} x \lim_{x\to1} x - 2\lim_{x\to1} x + \lim_{x\to1} 3} \qquad \text{(part 2)}$$

$$= \frac{3(1\cdot 1) - 5}{1\cdot 1\cdot 1 - 2\cdot 1 + 3} \qquad \text{(Prop.)}$$

$$= \frac{-2}{2}$$

$$= -1.$$

□

Image how a straight ε-δ proof of this claim would look!

Where did ε and δ go in the proofs in Examples 1 and 2? The answer is that they are hidden inside the proof of our limit theorem. A complete proof of the limit theorem might appear in math 112. To give an idea, though, and to introduce the important "$\varepsilon/2$-trick", we'll prove part 1 of the theorem.

The complete proof of part (1) of the Limit Theorem will appear in the next lecture. For now, we'll consider the motivation for the proof. We are given that $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist. Say $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$. We must show that

$$\lim_{x\to c}(f(x)+g(x)) = L + M.$$

Given $\varepsilon > 0$, we will want

$$|(f(x)+g(x)) - (L+M)| < \varepsilon,$$

when x is close to c. Since $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, we get to suppose that we can make both $|f(x) - L|$ and $|g(x) - M|$ small when x is close to c. However, note that

$$|(f(x) + g(x)) - (L + M)| = |(f(x) - L) + (g(x) - M)| \qquad (\star)$$

If $f(x) - L$ and $g(x) - M$ are small, then the above equality gives us hope that we can make $|(f(x) + g(x)) - (L + M)|$ small, too. Looking ahead, if we want $|(f(x) + g(x)) - (L + M)| < \varepsilon$, in light of $(\star)$ it seems reasonable reasonable to require $|f(x) - M|$ and $|g(x) - M|$ to both be less that $\varepsilon/2$. We'll put these ideas together in a rigorous proof next time.